GROUPS, ACTIONS AND VON NEUMANN ALGEBRAS LECTURE NOTES

CYRIL HOUDAYER

Abstract. These are the lecture notes of a graduate course given at the Université Paris-Sud (Orsay) in the Winter of 2016. In Section 1, we first review some preliminary background on C*-algebras. In Section 2, we introduce von Neumann algebras and prove some basic properties. In Section 3, we present two important classes of von Neumann algebras, namely group von Neumann algebras and Murray–von Neumann's group measure space constructions. In Section 4, we prove Connes's characterization of amenable tracial von Neumann algebras. Finally in Section 5, we prove Ozawa–Popa's strong solidity result for free group factors.

Contents

1. Preliminary background on C*-algebras and functional analysis

All the algebras we consider are always over the field $\mathbf{C}$ of complex numbers.

1.1. Introduction to C*-algebras.

1.1.1. *Definition and first properties.*

Definition 1.1. A C*-*algebra* A is a Banach algebra endowed with an involution $A \to A : a \mapsto a^*$ which satisfies the following relation:

$$\|a^*a\| = \|a\|^2, \forall a \in A.$$

If A admits a unit, we say that A is a *unital* C*-algebra. Denote by $\mathbf{B}(H)$ the Banach algebra of all bounded linear operators $T : H \to H$ endowed with the *supremum norm*:

$$\|T\|_\infty = \sup_{\|\xi\| \leq 1} \|T\xi\|.$$

Let $T \in \mathbf{B}(H)$. The *adjoint operator* T^* is defined by

$$\langle T\xi, \eta\rangle = \langle \xi, T^*\eta\rangle, \forall \xi, \eta \in H.$$

Examples 1.2. Here are examples of C^*-algebras.

(1) Norm closed $*$-subalgebras of $\mathbf{B}(H)$.
(2) The space of all complex-valued continuous functions $\mathrm{C}(X)$ over a compact topological space X endowed with the supremum norm given by $\|f\|_\infty = \sup_{x\in X}|f(x)|$. The involution is given by $f^*(x) = \overline{f(x)}$ for all $x \in X$.
(3) Let Γ be a countable discrete group and let $\lambda : \Gamma \to \mathcal{U}(\ell^2(\Gamma))$ be the *left regular representation* defined by $\lambda_g\delta_h = \delta_{gh}$ for all $g, h \in \Gamma$. The *reduced group* C^**-algebra* $\mathrm{C}^*_\lambda(\Gamma)$ is defined as the norm closure of the linear span of $\{\lambda_g : g \in \Gamma\}$.

From now on, to avoid any technical difficulties, we will always assume that all C^*-algebras are unital. For $a \in A$, the *spectrum* of a is defined as follows:

$$\sigma(a) := \{\lambda \in \mathbf{C} : a - \lambda 1 \text{ is not invertible}\}.$$

Proposition 1.3. *For all $a \in A$, $\sigma(a)$ is a nonempty compact subset of $\mathbf{C}$.*

Proof. It is clear that $\sigma(a)$ is closed. Moreover for all $|\lambda| > \|a\|$, $1 - \lambda^{-1}a$ is invertible with inverse $\sum_n \lambda^{-n}a^n$. It follows that $\sigma(a)$ is bounded by $\|a\|$, whence $\sigma(a)$ is compact.

By contradiction, assume that $\sigma(a)$ is the empty set. Then the function $\lambda \mapsto (a - \lambda 1)^{-1}$ is entire and vanishing at infinity. By Hahn–Banach and Liouville Theorems, we get that this function is zero everywhere. Thus $a^{-1} = 0$, which is a contradiction. Thus $\sigma(a)$ is nonempty and compact. □

Observe that the above proof works more generally for any unital Banach algebra. We have the following useful corollary.

Corollary 1.4. *Any unital Banach algebra A in which every nonzero element is invertible is isomorphic to $\mathbf{C}$.*

Proof. Let $x \in A$ and choose $\lambda \in \sigma(a)$. Since $x - \lambda 1$ is not invertible, we have $x - \lambda 1 = 0$. Thus $A = \mathbf{C}1$. □

Exercise 1.5. Show that $\sigma(ab) \cup \{0\} = \sigma(ba) \cup \{0\}$, for all $a, b \in A$.

Exercise 1.6. Let A be a unital abelian Banach algebra and $\mathfrak{m} \subset A$ a proper ideal, that is, $1 \notin \mathfrak{m}$. Show that

$$\inf\{\|1 - x\| : x \in \mathfrak{m}\} \geq 1.$$

Deduce that the closure of any proper ideal is still proper and any maximal proper ideal is closed.

The *spectral radius* is defined by

$$r(a) := \sup\{|\lambda| : \lambda \in \sigma(a)\}.$$

We have $r(a) \leq \|a\|$.

Proposition 1.7. *For all $a \in A$, the sequence $(\|a^n\|^{1/n})_n$ converges to $r(a)$.*

Proof. If $\lambda \in \sigma(a)$, then $\lambda^n \in \sigma(a^n)$. Thus $|\lambda| \leq \|a^n\|^{1/n}$, for all $n \in \mathbf{N}$. It follows that $|\lambda| \leq \liminf \|a^n\|^{1/n}$ and hence $r(a) \leq \liminf_n \|a^n\|^{1/n}$. Next, for $|z| < r(a)^{-1}$, $f : z \mapsto (1 - za)^{-1}$ is a holomorphic function which coincides with the power series $\sum_n z^n a^n$ when moreover $|z| < \|a\|^{-1}$. Observe that this power series represents f on the open disk with center 0 and radius $r(a)^{-1}$. However, this series cannot converge for $|z| > (\limsup \|a^n\|^{1/n})^{-1}$. Thus, we get that $\limsup \|a^n\|^{1/n} \leq r(a)$. □

In particular, if $a, b \in A$ are commuting elements, we have that

$$\begin{aligned} r(ab) &= \lim \|(ab)^n\|^{1/n} = \lim \|a^n b^n\|^{1/n} \\ &\leq \lim \|a^n\|^{1/n} \lim \|b^n\|^{1/n} \\ &= r(a)r(b). \end{aligned}$$

We say that a is *selfadjoint* if $a^* = a$; *normal* if $a^*a = aa^*$; *unitary* if $a^*a = aa^* = 1$. The group of unitaries is denoted by $\mathcal{U}(A)$. The subspace of selfadjoint elements in A is sometimes denoted by $\Re(A)$.

Proposition 1.8. *Let $a \in A$. The following are true.*

1. *If a is invertible, a^* is invertible and $(a^*)^{-1} = (a^{-1})^*$*
2. *a can be uniquely decomposed $a = x + \mathrm{i}y$, with x, y selfadjoint elements.*
3. *If a is a unitary then $\|a\| = 1$.*
4. *If a is normal then $\|a\| = r(a)$.*
5. *If $\mathcal{B}$ is another C^*-algebra and $\varphi : A \to \mathcal{B}$ is a $*$-homomorphism then $\|\varphi(a)\| \leq \|a\|$.*

Proof. We leave $(1), (2), (3)$ as an exercise. To prove (4), first assume that a is selfadjoint. One has $\|a^{2^n}\| = \|a\|^{2^n}$ for all $n \in \mathbf{N}$. Thus, $r(a) = \lim_n \|a^{2^n}\|^{2^{-n}} = \|a\|$. If a is normal, $\|a\|^2 = \|a^*a\| = r(a^*a) \leq r(a^*)r(a) \leq \|a^*\|\|a\| = \|a\|^2$, whence $r(a) = \|a\|$. To prove (5), let $a \in A$. Then

$$\|\varphi(a)\|^2 = \|\varphi(a)^*\varphi(a)\| = \|\varphi(a^*a)\| = r(\varphi(a^*a)) \leq r(a^*a) = \|a^*a\| = \|a\|^2.$$

□

Corollary 1.9. *Any onto $*$-isomorphism $\varphi : A \to B$ is isometric.*

1.1.2. *Continuous functional calculus.*

Lemma 1.10. *Let $\chi : A \to \mathbf{C}$ be a unital algebraic homomorphism. Then the following assertions hold true.*

1. *For all $a \in A$, $|\chi(a)| \leq \|a\|$.*
2. *For all $a \in \Re(A)$, $\chi(a) \in \mathbf{R}$.*
3. *For all $a \in A$, $\chi(a^*) = \overline{\chi(a)}$.*
4. *For all $a \in A$, $\chi(a^*a) \geq 0$.*
5. *For all $a \in \mathcal{U}(A)$, $|\chi(a)| = 1$.*

Proof. (1) For all $a \in A$, $\chi(a - \chi(a)1) = 0$, whence $a - \chi(a)1$ is not invertible. We get $\chi(a) \in \sigma(a)$ and so $|\chi(a)| \leq \|a\|$.

(2) Assume that $a \in A$ is selfadjoint. Let $t \in \mathbf{R}$.

$$|\chi(a + \mathrm{i}t)|^2 \leq \|a + \mathrm{i}t\|^2 = \|(a + \mathrm{i}t)^*(a + \mathrm{i}t)\| = \|(a - \mathrm{i}t)(a + \mathrm{i}t)\| \leq \|a\|^2 + t^2.$$

Write $\chi(a) = \alpha + \mathrm{i}\beta$. We then get

$$\|a\|^2 + t^2 \geq |\alpha + \mathrm{i}(\beta + t)|^2 = \alpha^2 + \beta^2 + 2\beta t + t^2.$$

It follows that $\|a\|^2 \geq \alpha^2 + \beta^2 + 2\beta t$ and thus $\beta = 0$.

Now (3) follows easily, while (4) and (5) are trivial. □

Corollary 1.11. *Every unital algebraic homomorphism $\chi : A \to \mathbf{C}$ is necessarily a $*$-homomorphism.*

For a unital abelian C*-algebra A, a unital algebraic homomorphism $\chi : A \to \mathbf{C}$ is simply called a *character*. We will denote by $\Omega := \Omega(A)$ the set of characters of A. Sometimes Ω is called the *spectrum* of A. Observe that if $\chi : A \to \mathbf{C}$ is a character, we have that $\chi \in A^*$ and $\|\chi\|_{A^*} = 1$. One checks that Ω is closed for the $\sigma(A^*, A)$-topology and thus compact by Banach-Alaoglu Theorem. The *Gelfand Transform* $\gamma : A \to \mathrm{C}(\Omega)$ is defined by $\gamma(a)(\chi) = \chi(a)$.

Theorem 1.12. *The Gelfand Transform $\gamma : A \to \mathrm{C}(\Omega)$ is an onto $*$-isomorphism. Moreover $\sigma(a) = \{\chi(a) : \chi \in \Omega\}$, for all $a \in A$.*

Proof. Let $a \in A$. We have already shown that $\{\chi(a) : \chi \in \Omega\} \subset \sigma(a)$. If $\lambda \in \sigma(a)$, then $a - \lambda 1$ is not invertible. It is thus contained in a maximal proper ideal $\mathfrak{m}$, which is closed by Exercise 1.6. Observe that the Banach algebra $A/\mathfrak{m}$ is a division ring and so is isomorphic to $\mathbf{C}$. Whence there exists $\chi \in \Omega$ such that $\chi(a - \lambda 1) = 0$, that is, $\chi(a) = \lambda$. Therefore $\sigma(a) = \{\chi(a) : \chi \in \Omega\}$.

It is then clear that γ is a $*$-isomorphism and is isometric. Indeed, for all $a \in A$, we have

$$\|\gamma(a)\|_\infty^2 = \|\gamma(a)^*\gamma(a)\|_\infty = \|\gamma(a^*a)\|_\infty = r(a^*a) = \|a^*a\| = \|a\|^2.$$

Thus, $\gamma(A)$ is a closed $*$-subalgebra of $\mathrm{C}(\Omega)$. It remains to prove that γ is onto. Observe that $\gamma(A)$ separates points: for all $\chi \neq \chi'$, there exists $a \in A$ such that $\chi(a) \neq \chi'(a)$, that is, $\gamma(a)(\chi) \neq \gamma(a)(\chi')$. By Stone–Weierstrass's Theorem, $\gamma(A)$ is dense in $\mathrm{C}(\Omega)$. Therefore $\gamma(A) = \mathrm{C}(\Omega)$. □

Corollary 1.13. *If $a \in A$ is a unitary, then $\sigma(a) \subset \mathbf{T}$. If $a \in A$ is selfadjoint, then $\sigma(a) \subset \mathbf{R}$.*

Theorem 1.14 (Continuous functional calculus)**.** *Let A be a unital* C*-*algebra and $b \in A$ be a normal element. Denote by B the abelian* C*-*algebra generated by b. There exists a unique onto $*$-isomorphism $\Phi : \mathrm{C}(\sigma(b)) \to B$ such that $\sigma(\Phi(f)) = f(\sigma(b))$.*

We will simply denote $\Phi(f)$ by $f(b)$. Observe that, in particular, we have that $\|f(b)\| = \|f\|_\infty$.

Proof. Let Ω be the set of characters of $\mathcal{B}$. Define the continuous function $\psi : \Omega \to \sigma(b)$ by $\psi(\chi) = \chi(b)$. We have seen before that ψ is onto. Assume now that $\psi(\chi) = \psi(\chi')$, that is, $\chi(b) = \chi'(b)$. It follows that $\chi(p(b, b^*)) = \chi'(p(b, b^*))$ for all polynomials p. Since b generates $\mathcal{B}$, we get that $\chi = \chi'$ by Stone–Weierstrass's Theorem. Therefore ψ is a homeomorphism. Then $\widehat{\psi} : \mathrm{C}(\Omega) \to \mathrm{C}(\sigma(b))$ defined by $\widehat{\psi}(f) = f \circ \psi$ is an onto $*$-isomorphism. Now the $*$-isomorphism $\Phi = \gamma^{-1} \circ \widehat{\psi}^{-1} : \mathrm{C}(\sigma(b)) \to \mathcal{B}$ does the job. □

1.1.3. *The Gelfand–Naimark–Segal construction.*

Definition 1.15. An element $a \in A$ is *positive* if $a = a^*$ and $\sigma(a) \subset \mathbf{R}_+$. We will denote $a \geq 0$. The set of positive elements in A will be also denoted by A_+.

An element $a \in A$ is *negative* if $-a$ is positive. The set of negative elements in A will be denoted by A_-. For selfadjoint elements $a, b \in A$, we write $a \leq b$ when $b - a \in A_+$.

Proposition 1.16. *Let A be a unital* C*-*algebra and let $a \in A$ be a selfadjoint element. There exists a unique pair (h, k) of positive elements in A such that $a = h - k$ and $hk = kh = 0$.*

Proof. Define the continuous functions $f(t) = \max(t, 0)$ and $g(t) = \max(-t, 0)$ so that $f(t) - g(t) = t$, $f(t) \geq 0$, $g(t) \geq 0$ and $f(t)g(t) = 0$. By continuous functional calculus, we have $a = f(a) - g(a)$, $f(a) \geq 0$, $g(a) \geq 0$ and $f(a)g(a) = g(a)f(a) = 0$. We have proven the existence of the decomposition. To prove the uniqueness, assume that $a = u - v$ for some $u, v \in A_+$ such that $uv = vu = 0$. It is not hard to see that u and v commute with a so that the C*-algebra

$\mathrm{C}^*(a,u,v)$ is abelian. There exists some compact space X such that $\mathrm{C}^*(a,u,v) = \mathrm{C}(X)$. It only remains to prove the uniqueness of the decomposition for continuous functions on X which is fairly easy. □

Exercise 1.17. Let A be a unital C*-algebra.

- Let $a \in A_+$ and $n \geq 1$. Show that there exists a unique $b \in A_+$ such that $a = b^n$.
- Let $a \in A$ selfadjoint. Show that $a \geq 0$ if and only if $\|t - a\| \leq t$ for some $t \geq \|a\|$. Deduce that if $a, b \geq 0$, then $a + b \geq 0$.

Proposition 1.18. *Let A be a unital* C*-*algebra and $a \in A$. The following are equivalent:*

- $a \geq 0$.
- *There exists $b \in A$ such that $a = b^*b$.*

Proof. Assume that $a = b^*b$ and write $a = h - k$ as in Proposition 1.16. We want to show that $k = 0$. Set $bk^{1/2} = \alpha + \mathrm{i}\beta$, with α, β selfadjoint elements in A. On the one hand, we have

$$(bk^{1/2})^*(bk^{1/2}) = k^{1/2}b^*bk^{1/2} = k^{1/2}(h-k)k^{1/2} = -k^2 \leq 0,$$

since $hk = kh = 0$. On the other hand,

$$(bk^{1/2})^*(bk^{1/2}) = (\alpha + \mathrm{i}\beta)^*(\alpha + \mathrm{i}\beta) = \alpha^2 + \beta^2 + \mathrm{i}(\alpha\beta - \beta\alpha).$$

Thus $\mathrm{i}(\alpha\beta - \beta\alpha) = -k^2 - \alpha^2 - \beta^2 \leq 0$. Observe that $\sigma((bk^{1/2})^*(bk^{1/2}))$ and $\sigma((bk^{1/2})(bk^{1/2})^*)$ only differ by 0 (see Exercise 1.5). Thus $(bk^{1/2})(bk^{1/2})^* = -c$ with $c \in A_+$. We get $-c = \alpha^2 + \beta^2 + \mathrm{i}(\beta\alpha - \alpha\beta)$, so that $\mathrm{i}(\alpha\beta - \beta\alpha) = c + \alpha^2 + \beta^2 \geq 0$. Therefore $\mathrm{i}(\alpha\beta - \beta\alpha) \in A_+ \cap A_-$ and so $\mathrm{i}(\alpha\beta - \beta\alpha) = 0$. This implies that $-k^2 = (bk^{1/2})^*(bk^{1/2}) = \alpha^2 + \beta^2 \in A_+ \cap A_-$ and thus $k = 0$. □

Exercise 1.19. Show that for all $a \in A$, $a^*a \leq \|a\|^2 1$.

Definition 1.20. A *state* $\varphi : A \to \mathbf{C}$ is a positive linear functional ($\varphi(a) \geq 0$ for all $a \geq 0$) such that $\varphi(1) = 1$. The state space of A is denoted by $\Sigma(A)$. A state φ is *faithful* if $\varphi(a^*a) > 0$ for all $a \neq 0$.

Example 1.21. Let (π, H, ξ) be a unital $*$-representation of A together with a unit vector. The linear functional $a \mapsto \langle \pi(a)\xi, \xi\rangle$ defines a state on A. We will prove that every state on a unital C*-algebra arises this way.

Proposition 1.22. *Let $\varphi : A \to \mathbf{C}$ be a positive linear functional. The following hold true.*

1. *For all $a, b \in A$, $|\varphi(b^*a)|^2 \leq \varphi(a^*a)\varphi(b^*b)$*
2. *φ is bounded and $\|\varphi\| = \varphi(1)$. In particular, if φ is a state then $\|\varphi\| = 1$.*

Proof. Observe that $(a, b) \mapsto \varphi(b^*a)$ defines a semi-inner product on A. Then (1) follows from the Cauchy–Schwarz Inequality. For (2), observe that since $a^*a \leq \|a\|^2 1$, we have $|\varphi(a)|^2 \leq \varphi(1)\varphi(a^*a) \leq \varphi(1)^2\|a\|^2$. It follows that $\|\varphi\| = \varphi(1)$. □

Example 1.23. Let X be a compact space. Any probability measure μ on X gives rise to a state φ on $\mathrm{C}(X)$ by $\varphi(f) = \int_X f \, \mathrm{d}\mu$. By Riesz Representation Theorem, any state on $\mathrm{C}(X)$ arises this way.

Exercise 1.24. Let A be a unital C*-algebra and let $\varphi : A \to \mathbf{C}$ be a bounded linear functional with $\|\varphi\| = \varphi(1)$. Show that φ is positive. Deduce that if $\mathcal{B} \subset A$ is a unital C*-subalgebra, then any state on $\mathcal{B}$ has an extension on A.

Theorem 1.25 (GNS construction)**.** *Let A be a unital* C^**-algebra.*

(1) For every state φ on A, there exists a cyclic $*$-representation (π_φ, H_φ) together with a unit vector $\xi_\varphi \in H_\varphi$ such that $\varphi(a) = \langle \pi_\varphi(a)\xi_\varphi, \xi_\varphi\rangle$, for all $a \in A$.
(2) If (π, H) is a cyclic $*$-representation with unit cyclic vector $\xi \in H$ and φ is the state defined by $\varphi(a) = \langle \pi(a)\xi, \xi\rangle$, then $\pi \cong \pi_\varphi$.

Proof. (1) Let φ be a state on A. Define the following semi-inner product $\langle a, b\rangle_\varphi = \varphi(b^* a)$ on A. After separation and completion, promote $(A, \langle \cdot, \cdot\rangle_\varphi)$ to a genuine Hilbert space H_φ. Denote by $a^\bullet$ the image of $a \in A$ in H_φ. One checks that $\pi_\varphi(a)b^\bullet = (ab)^\bullet$ defines a cyclic $*$-representation with unit cyclic vector $\xi_\varphi = 1^\bullet$. Indeed, for all $a, b \in A$, we have

$$\begin{aligned}
\|\pi_\varphi(a)b^\bullet\|_\varphi^2 &= \langle \pi_\varphi(a)b^\bullet, \pi_\varphi(a)b^\bullet\rangle_\varphi \\
&= \langle \pi_\varphi(a^*a)b^\bullet, b^\bullet\rangle_\varphi \\
&= \varphi(b^*\, a^* a\, b) \\
&\leq \|a\|^2 \varphi(b^* b) \\
&= \|a\|^2 \|b^\bullet\|_\varphi^2
\end{aligned}$$

and hence $\pi_\varphi(a) \in \mathbf{B}(H_\varphi)$ is well-defined. For all $a \in A$, we moreover have

$$\langle \pi_\varphi(a)\xi_\varphi, \xi_\varphi\rangle_\varphi = \langle a^\bullet, 1^\bullet\rangle_\varphi = \varphi(a).$$

We leave (2) as an exercise. ◻

Corollary 1.26. *Every unital* C^**-algebra admits a unital faithful $*$-representation (π, H). Moreover, H can be chosen to be separable if A is separable.*

Proof. Let $S \subset \Sigma(A)$ be a weak*-dense subset. Note that if A is separable, S can be taken countable. Define $\pi = \bigoplus_{\varphi \in S} \pi_\varphi$. Assume that $\pi(a) = 0$, that is, $\pi(a^*a) = 0$. We get $\varphi(a^*a) = 0$ for all $\varphi \in S$. By density, we get $\varphi(a^*a) = 0$, for all $\varphi \in \Sigma(A)$.

Let now μ be a probability measure on $X := \sigma(a^*a)$ and define the state $\psi(f(a^*a)) = \int_X f \, \mathrm{d}\mu$ for all $f \in \mathrm{C}(X)$. Extend ψ to φ on A. We have

$$\int_X t \, \mathrm{d}\mu(t) = \psi(a^*a) = \varphi(a^*a) = 0.$$

It follows that $X = \{0\}$ and so $a = 0$. ◻

1.2. Topologies on $\mathbf{B}(H)$.

Definition 1.27. Let H be a complex Hilbert space.

- The *strong operator topology* (SOT) on $\mathbf{B}(H)$ is defined by the following family of open neighbourhoods: for $S \in \mathbf{B}(H)$, $\varepsilon > 0$, $\xi_1, \dots, \xi_n \in H$, define
$$\mathcal{U}(S, \varepsilon, \xi_i) := \{T \in \mathbf{B}(H) : \|(T - S)\xi_i\| < \varepsilon, \forall 1 \leq i \leq n\}.$$
- The *weak operator topology* (WOT) on $\mathbf{B}(H)$ is defined by the following family of open neighbourhoods: for $S \in \mathbf{B}(H)$, $\varepsilon > 0$, $\xi_1, \dots, \xi_n, \eta_1, \dots, \eta_n \in H$, define
$$\mathcal{V}(S, \varepsilon, \xi_i, \eta_i) := \{T \in \mathbf{B}(H) : |\langle (T - S)\xi_i, \eta_i\rangle| < \varepsilon, \forall 1 \leq i \leq n\}.$$

The strong operator topology is always stronger than the weak operator topology. It is strictly stronger when H is infinite dimensional.

Theorem 1.28. *Let $\mathcal{C} \subset \mathbf{B}(H)$ be a nonempty convex subset. Then the strong operator closure and the weak operator closure of $\mathcal{C}$ coincide.*

Proof. Assume T is in the weak operator closure of $\mathcal{C}$. Let $\xi_1, \dots, \xi_n \in H$. Let $K = H \oplus \dots \oplus H$ be the n-fold direct sum of H with itself. Define the $*$-isomorphism $\rho : \mathbf{B}(H) \to \mathbf{B}(K)$ by $\rho(T)(\eta_1, \dots, \eta_n) = (T\eta_1, \dots, T\eta_n)$. Let $\xi = (\xi_1, \dots, \xi_n) \in K$. It is clear that $\rho(\mathcal{C})$ is a convex subset of $\mathbf{B}(K)$. Since $\rho(T)$ is in the weak operator closure of $\rho(\mathcal{C})$, $\rho(T)\xi$ is in the weak closure of $\rho(\mathcal{C})\xi$. Since $\rho(\mathcal{C})\xi \subset K$ is convex, the Hahn–Banach Separation Theorem implies that $\rho(T)\xi$ is also in the norm closure of $\rho(\mathcal{C})\xi$. For $\varepsilon > 0$, there exists $S \in \mathcal{C}$ such that $\|S\xi_i - T\xi_i\| < \varepsilon$, for all $1 \leq i \leq n$. This shows that T is in the strong operator closure of $\mathcal{C}$. □

Proposition 1.29. *Let $V \subset \mathbf{B}(H)$ be a weakly closed subspace and $\varphi : V \to \mathbf{C}$ a linear functional. The following are equivalent.*

(1) There exist $\xi_1, \dots, \xi_n, \eta_1, \dots, \eta_n \in H$ such that

$$\varphi(T) = \sum_{i=1}^{n} \langle T\xi_i, \eta_i \rangle, \forall T \in V.$$

(2) φ is strongly continuous.
(3) φ is weakly continuous.

Proof. (1) $\Rightarrow$ (2) is clear. For (2) $\Rightarrow$ (1), let $\varepsilon > 0$ and $\xi_1, \dots, \xi_n \in H$ such that $|\varphi(x)| \leq 1$ for all $x \in \mathcal{U}(0, \varepsilon, \xi_i)$. It follows that $|\varphi(x)| \leq \frac{1}{\varepsilon}\sqrt{\sum_i \|x\xi_i\|^2}$ for all $x \in V$. Let $\xi = (\xi_1, \dots, \xi_n) \in \ell^2_n \otimes H$ and $\mathcal{K} = \overline{(1 \otimes V)\xi} \subset \ell^2_n \otimes H$. Define the continuous linear functional $\psi : \mathcal{K} \to \mathbf{C}$ by $\psi((1 \otimes x)\xi) = \varphi(x)$ for all $x \in V$. By Representation Theorem, there exists $\eta \in \mathcal{K}$ such that $\varphi(x) = \langle (1 \otimes x)\xi, \eta \rangle$ for all $x \in V$.

Notice that φ is continuous if and only if $\ker \varphi$ is closed. Since $\ker \varphi \subset \mathbf{B}(H)$ is a nonempty convex subset, the equivalence between (2) and (3) follows from Theorem 1.28. □

Theorem 1.30. *The unit ball $(\mathbf{B}(H))_1$ is weakly compact.*

Proof. Denote by $\mathbf{D}_{\xi,\eta}$ the closed unit disk in $\mathbf{C}$ of center 0 and radius $\|\xi\|\|\eta\|$. The map $(\mathbf{B}(H))_1 \ni T \mapsto (\langle T\xi, \eta \rangle)_{\xi,\eta \in H} \in \prod_{\xi,\eta \in H} \mathbf{D}_{\xi,\eta}$ is a homeomorphism from $(\mathbf{B}(H))_1$, endowed with the weak operator topology onto its image X. Note that $\prod_{\xi,\eta \in H} \mathbf{D}_{\xi,\eta}$ is compact for the product topology by Tychonoff's Theorem. It remains to show that the image X is closed.

Let $\alpha = (\alpha_{\xi,\eta}) \in \overline{X}$. There exists a net (S_i) of elements in $(\mathbf{B}(H))_1$ such that $\langle S_i\xi, \eta \rangle \to \alpha_{\xi,\eta}$, for all $\xi, \eta \in H$. We get that $H \times H \ni (\xi, \eta) \mapsto \alpha_{\xi,\eta} \in \mathbf{C}$ is a conjugate-bilinear form such that $|\alpha_{\xi,\eta}| \leq \|\xi\|\|\eta\|$, for all $\xi, \eta \in H$. By Riesz Representation Theorem for conjugate-bilinear forms, there exists $T \in (\mathbf{B}(H))_1$ such that $\alpha_{\xi,\eta} = \langle T\xi, \eta \rangle$, for all $\xi, \eta \in H$. □

Proposition 1.31. *Let (T_i) be an increasing net of selfadjoint operators such that $T_i \leq C1$, for all $i \in I$. Then (T_i) has a limit with respect to the strong operator topology. Moreover, for all $S \in \mathbf{B}(H)$ such that $T_i \leq S$, for all $i \in I$, we have that $\lim T_i \leq S$. We denote $\lim T_i = \sup T_i$.*

Proof. Without loss of generality, we may assume that (T_i) is bounded from below as well, that is, $-C1 \leq T_i \leq C1$, for all $i \in I$. By weak compactness of the unit ball, we can find a subnet (T_j) which converges weakly to some selfadjoint operator $T \in \mathbf{B}(H)$.

Let $i \in I$. For all $j \geq i$, $\xi \in H$, $\langle T_j\xi, \xi \rangle \geq \langle T_i\xi, \xi \rangle$ so that $\langle T\xi, \xi \rangle = \lim_j \langle T_j\xi, \xi \rangle \geq \langle T_i\xi, \xi \rangle$. Thus, for all $i \geq j$, $0 \leq T - T_i \leq T - T_j$ so that

$$\|(T - T_i)^{1/2}\xi\|^2 \leq \langle (T - T_i)\xi, \xi \rangle \leq \langle (T - T_j)\xi, \xi \rangle \to 0.$$

We have that $(T-T_i)^{1/2} \to 0$ strongly. Finally, strong continuity of multiplication on uniformly bounded sets yields $(T-T_i) \to 0$ strongly.

We have already seen that $T_i \leq T$, for all $i \in I$. Assume now that $T_i \leq S$, for all $i \in I$. Since $T_i \to T$ strongly, we have that $T_i \to T$ weakly, whence for all $\xi \in H$, $\langle T\xi, \xi\rangle = \lim \langle T_i \xi, \xi\rangle \leq \langle S\xi, \xi\rangle$. □

Definition 1.32. Let H be a complex Hilbert space.

- The *ultrastrong operator topology* on $\mathbf{B}(H)$ is defined by the following family of open neighbourhoods: for $S \in \mathbf{B}(H)$, $\varepsilon > 0$, $(\xi_n) \in \ell^2(\mathbf{N}, H)$, define

$$\mathcal{U}(S, \varepsilon, (\xi_n)) := \left\{ T \in \mathbf{B}(H) : \sum_n \|(T-S)\xi_n\|^2 < \varepsilon \right\}.$$

- The *ultraweak operator topology* on $\mathbf{B}(H)$ is defined by the following family of open neighbourhoods: for $S \in \mathbf{B}(H)$, $\varepsilon > 0$, $(\xi_n), (\eta_n) \in \ell^2(\mathbf{N}, H)$, define

$$\mathcal{V}(S, \varepsilon, (\xi_n), (\eta_n)) := \left\{ T \in \mathbf{B}(H) : \left| \sum \langle (T-S)\xi_n, \eta_n \rangle \right| < \varepsilon \right\}.$$

Exercise 1.33. Show that on uniformly bounded sets, weak (resp. strong) and ultraweak (resp. ultrastrong) topologies coincide.

Proposition 1.34. *Let $\varphi : \mathbf{B}(H) \to \mathbf{C}$ be a linear form. The following are equivalent.*

(1) There exists $(\xi_n), (\eta_n) \in \ell^2(\mathbf{N}, H)$ such that

$$\varphi(T) = \sum_n \langle T\xi_n, \eta_n \rangle, \forall T \in \mathbf{B}(H).$$

(2) φ is ultrastrongly continuous.
(3) φ is ultraweakly continuous.
(4) φ is strongly continuous on $(\mathbf{B}(H))_1$.
(5) φ is weakly continuous on $(\mathbf{B}(H))_1$.

Proof. The proof is analogous to Proposition 1.29, so we leave it as an exercise. □

2. Introduction to von Neumann algebras

2.1. **Definition and first examples of von Neumann algebras.** For a nonempty subset $\mathcal{S} \subset \mathbf{B}(H)$, the *commutant* of $\mathcal{S}$ is defined by

$$\mathcal{S}' := \{T \in \mathbf{B}(H) : ST = TS, \forall S \in \mathcal{S}\}.$$

It is easy to see that one always has $\mathcal{S} \subset \mathcal{S}''$. Moreover, if $\mathcal{S}$ is stable under the adjoint operation, then $\mathcal{S}'$ is a unital $*$-algebra.

Theorem 2.1 (Bicommutant Theorem)**.** *Let $M \subset \mathbf{B}(H)$ be a unital $*$-subalgebra. The following are equivalent.*

(1) $M = M''$.
(2) M is strongly closed.
(3) M is weakly closed.

Proof. (1) $\Rightarrow$ (2). Let $(x_i)_{i\in I}$ be a net in M such that $x_i \to x$ strongly. Since $x_i T = T x_i$ for all $i \in I$ and $T \in M'$, by passing to the limit we get $xT = Tx$, for all $T \in M'$. Thus $x \in M$.

(2) $\Rightarrow$ (1). Let $x \in M''$ and $\xi_1, \dots, \xi_n \in H$. Let

$$\mathcal{U}(x, \varepsilon, \xi_i) := \{y \in \mathbf{B}(H) : \|x\xi_i - y\xi_i\| < \varepsilon, \forall i = 1, \dots, n\}$$

be a strong neighborhood of x in $\mathbf{B}(H)$. Let $K = \ell^2_n \otimes H$ and observe that $\mathbf{B}(K) = \mathbf{M}_n(\mathbf{C}) \otimes \mathbf{B}(H)$. Let $\eta = (\xi_1, \dots, \xi_n) \in K$. Define $V = \overline{(1 \otimes M)\eta} \subset K$. Denote by $P_V \in \mathbf{B}(K)$ the corresponding orthogonal projection. Since $(1 \otimes a)P_V = P_V(1 \otimes a)$, $\forall a \in M$, it follows that $1 \otimes x$ commutes with P_V, since $x \in M''$. Thus $(1 \otimes x)\eta \in V$ and we can find $y \in M$ such that $\|(1 \otimes x)\eta - (1 \otimes y)\eta\| < \varepsilon$, so in particular $y \in \mathcal{U}(x, \varepsilon, \xi_i)$. Then M'' is contained in the strong closure of M and hence $M = M''$.

Since $M \subset \mathbf{B}(H)$ is convex, (2) $\Leftrightarrow$ (3) follows from Theorem 1.28. $\square$

Definition 2.2. A *von Neumann algebra* M is a unital $*$-subalgebra of $\mathbf{B}(H)$ which satisfies one of the equivalent conditions of Theorem 2.1.

Definition 2.3. Let $M \subset \mathbf{B}(H)$ be a von Neumann algebra. We say that

- $p \in M$ is a *projection* if $p = p^* = p^2$.
- $v \in M$ is an *isometry* if $v^*v = 1$.
- $u \in M$ is a *partial isometry* if u^*u is a projection.

Observe that if u^*u is a projection, then uu^* is a projection as well. The set of projections of M will be denoted by $\mathcal{P}(M)$. If $K \subset H$ is a closed subspace, we denote by $[K] \in \mathbf{B}(H)$ the orthogonal projection $[K] : H \to K$.

We will always assume that M is *σ-finite*, that is, any family $(p_i)_{i\in I}$ of pairwise orthogonal projections in M is (at most) countable.

Exercise 2.4. Let M be a von Neumann algebra. The closed subspace $K \subset H$ is u-invariant for all $u \in \mathcal{U}(M)$ if and only if $[K] \in M'$.

If $(p_i)_{i\in I}$ is a family of projections, we denote by

$$\bigvee_{i\in I} p_i = \left[\overline{\sum_{i\in I} \operatorname{ran}(p_i)}\right]$$

$$\bigwedge_{i\in I} p_i = \left[\bigcap_{i\in I} \operatorname{ran}(p_i)\right].$$

If $p \in \mathbf{B}(H)$ is a projection, write $p^\perp = 1 - p$. It is easy to check that $(\bigvee_{i\in I} p_i)^\perp = \bigwedge_{i\in I} p_i^\perp$.

Proposition 2.5. *Let $M \subset \mathbf{B}(H)$ be a von Neumann algebra. Then $\mathcal{P}(M)$ is a complete lattice.*

Proof. Let $(p_i)_{i\in I}$ be a family of projections in M. Since $M = (M')'$, we have that $\operatorname{ran}(p_i)$ is u-invariant for all $u \in \mathcal{U}(M')$ and all $i \in I$. Thus $\overline{\sum_{i\in I} \operatorname{ran}(p_i)}$ is u-invariant for all $u \in \mathcal{U}(M')$, whence $\bigvee_{i\in I} p_i \in M$. Moreover $\bigwedge_{i\in I} p_i = (\bigvee p_i^\perp)^\perp \in M$. $\square$

Theorem 2.6 (Polar decomposition)**.** *Let $T \in B(H)$. Then T can be written $T = U|T|$ where $U \in \mathbf{B}(H)$ is a partial isometry with initial support $\overline{\operatorname{ran}(T^*)}$ and final support $\overline{\operatorname{ran}(T)}$. Moreover, if $T = VS$ with $S \geq 0$ and V a partial isometry such that $V^*V = \overline{\operatorname{ran}(S)}$, then $S = |T|$ and $V = U$.*

Proof. Observe that $\ker(T) = \ker(T^*T) = \ker(|T|)$ so that $\overline{\mathrm{ran}(T^*)} = \ker(T)^\perp = \ker(|T|)^\perp = \overline{\mathrm{ran}(|T|)}$. Define $U\eta = 0$ for $\eta \in \mathrm{ran}(|T|)^\perp$ and $U|T|\xi = T\xi$, for all $\xi \in H$. One checks that $U \in \mathbf{B}(H)$ is a well-defined partial isometry such that $U^*U = [\overline{\mathrm{ran}(T^*)}]$, $UU^* = [\overline{\mathrm{ran}(T)}]$ and $T = U|T|$.

Assume now that $T = VS$ with $S \geq 0$ and $V^*V = \overline{\mathrm{ran}(S)}$. Then $T^*T = SV^*VS = S^2$. Thus $S = (T^*T)^{1/2} = |T|$. The formula $T = V|T|$ clearly shows that $V = U$. □

The first important example of von Neumann algebras we discuss comes from *measure theory.* Let (X, μ) be a standard probability space. Define the unital $*$-representation $\pi : \mathrm{L}^\infty(X, \mu) \to \mathbf{B}(\mathrm{L}^2(X, \mu))$ given by multiplication: $(\pi(f)\xi)(x) = f(x)\xi(x)$ for all $f \in \mathrm{L}^\infty(X, \mu)$ and all $\xi \in \mathrm{L}^2(X, \mu)$. Since π is a C*-algebraic isomorphism, we will identify $f \in \mathrm{L}^\infty(X, \mu)$ with its image $\pi(f) \in \mathbf{B}(\mathrm{L}^2(X, \mu))$. From now on, we will simply denote $\mathrm{L}^\infty(X, \mu)$ by $\mathrm{L}^\infty(X)$.

Proposition 2.7. *We have $\mathrm{L}^\infty(X)' \cap \mathbf{B}(\mathrm{L}^2(X, \mu)) = \mathrm{L}^\infty(X)$, that is, $\mathrm{L}^\infty(X)$ is maximal abelian in $\mathbf{B}(\mathrm{L}^2(X, \mu))$. In particular, $\mathrm{L}^\infty(X)$ is a von Neumann algebra.*

Proof. Let $T \in \mathrm{L}^\infty(X)' \cap \mathbf{B}(\mathrm{L}^2(X, \mu))$ and denote $f = T\mathbf{1}_X \in \mathrm{L}^2(X, \mu)$. For all $\xi \in \mathrm{L}^\infty(X) \subset \mathrm{L}^2(X, \mu)$, we have

$$T\xi = T\xi\, \mathbf{1}_X = \xi T\, \mathbf{1}_X = \xi f = f\xi.$$

For every $n \geq 1$, put $\mathcal{U}_n := \left\{x \in X : |f(x)| \geq \|T\|_\infty + \frac{1}{n}\right\}$. We have

$$\left(\|T\|_\infty + \frac{1}{n}\right) \mu(\mathcal{U}_n)^{1/2} \leq \|f\mathbf{1}_{\mathcal{U}_n}\|_2 = \|T\mathbf{1}_{\mathcal{U}_n}\|_2 \leq \|T\|_\infty\, \mu(\mathcal{U}_n)^{1/2},$$

hence $\mu(\mathcal{U}_n) = 0$ for every $n \geq 1$. This implies that $\|f\|_\infty \leq \|T\|_\infty$ and so $T = f$. □

The von Neumann algebra $M = \mathrm{L}^\infty(X)$ comes equipped with the faithful trace τ_μ given by integration against the probability measure μ,

$$\tau_\mu(f) = \int_X f \,\mathrm{d}\mu, \forall f \in \mathrm{L}^\infty(X).$$

2.2. **The predual.** Let M be a von Neumann algebra. Denote by $M_* \subset M^*$ the subspace of all ultraweakly continuous functionals on M. Recall the following fact.

Proposition 2.8. *We have that M_* is a closed subspace of M^*. Therefore, $(M_*, \|\cdot\|)$ is a Banach space.*

Proof. Let $\varphi \in M^*$ and $(\varphi_i)_{i\in I}$ be a net in M_* such that $\lim \|\varphi - \varphi_i\| = 0$. We have to show that φ is strongly continuous on $(M)_1$. Let $(x_j)_{j\in J}$ be a net in $(M)_1$ such that $x_j \to x$ strongly.

$$\begin{aligned} |\varphi(x) - \varphi(x_j)| &\leq |\varphi(x) - \varphi_i(x)| + |\varphi_i(x) - \varphi_i(x_j)| + |\varphi_i(x_j) - \varphi(x_j)| \\ &\leq 2\|\varphi - \varphi_i\| + |\varphi_i(x) - \varphi_i(x_j)|. \end{aligned}$$

Let $\varepsilon > 0$. Choose $i \in I$ such that $\|\varphi - \varphi_i\| \leq \varepsilon/3$. Since φ_i is ultraweakly continuous, choose $j_0 \in J$ such that for all $j \geq j_0$, $|\varphi_i(x) - \varphi_i(x_j)| \leq \varepsilon/3$. We get $|\varphi(x) - \varphi(x_j)| \leq \varepsilon$, for all $j \geq j_0$. □

Theorem 2.9. *Let M be any von Neumann algebra. The map $\Phi : M \to (M_*)^*$ defined by $\Phi(x)(\varphi) = \varphi(x)$ is an onto isometric linear map. Moreover, under the identification $M = (M_*)^*$, the ultraweak topology on M and the weak* topology on $(M_*)^*$ coincide.*

Proof. Assume $M \subset \mathbf{B}(H)$. For all $x \in M$, we have

$$\|x\|_\infty = \sup\left\{|\langle x\xi, \eta\rangle| : \xi, \eta \in H, \|\xi\| \leq 1, \|\eta\| \leq 1\right\}.$$

Put $\omega_{\xi,\eta} = \langle \cdot\, \xi, \eta\rangle$. Since $\omega_{\xi,\eta}|_M \in (M_*)_1$ for all $\xi, \eta \in H$ such that $\|\xi\| \leq 1$, $\|\eta\| \leq 1$, it follows that $\|x\|_\infty = \sup\left\{|\varphi(x)| : \varphi \in (M_*)_1\right\}$. Therefore Φ is an isometric embedding. It remains to show that Φ is onto.

Let $L \in (M_*)^*$. Define the bounded conjugate-bilinear form b on $H \times H$ by $b(\xi, \eta) = L(\omega_{\xi,\eta}|_M)$. By Riesz Representation Theorem for conjugate-bilinear forms, let $T \in \mathbf{B}(H)$ be the unique bounded operator such that $b(\xi, \eta) = \langle T\xi, \eta\rangle$ for all $\xi, \eta \in H$. Let $S \in M'$ be a selfadjoint element. For all $x \in M$, we have $\omega_{S\xi,\eta}(x) = \langle xS\xi, \eta\rangle = \langle Sx\xi, \eta\rangle = \langle x\xi, S\eta\rangle = \omega_{\xi,S\eta}(x)$ so that $\omega_{S\xi,\eta} = \omega_{\xi,S\eta}$. We obtain

$$\langle TS\xi, \eta\rangle = b(S\xi, \eta) = L(\omega_{S\xi,\eta}|_M) = L(\omega_{\xi,S\eta}|_M) = b(\xi, S\eta) = \langle ST\xi, \eta\rangle.$$

Therefore $T \in M'' = M$ by the Bicommutant Theorem. We have

$$\omega_{\xi,\eta}(T) = \langle T\xi, \eta\rangle = b(\xi, \eta) = L(\omega_{\xi,\eta}|_M).$$

Since any $\varphi \in M_*$ can be written $\varphi = \sum_n \omega_{\xi_n,\eta_n}|_M$ for some $(\xi_n), (\eta_n) \in \ell^2(\mathbf{N}, H)$ (see Proposition 1.34) and since L is continuous, we get $\varphi(T) = L(\varphi)$, for all $\varphi \in M_*$. Thus $L = \Phi(T)$. □

Definition 2.10. Let M and N be any von Neumann algebras. A positive linear map $\pi : M \to N$ is *normal* if for every uniformly bounded increasing net of selfadjoint elements $(x_i)_{i\in I}$ in M, we have

$$\pi\left(\sup_{i\in I} x_i\right) = \sup_{i\in I} \pi(x_i).$$

We have the following characterization of normal states.

Theorem 2.11. *Let M be a von Neumann algebra together with a state $\varphi \in M^*$. The following are equivalent.*

(1) φ is normal.
(2) Whenever $(p_i)_{i\in I}$ is a family of pairwise orthogonal projections in M, we have

$$\varphi\left(\sum_{i\in I} p_i\right) = \sum_{i\in I} \varphi(p_i).$$

(3) φ is ultraweakly continuous.

Proof. (1) ⇒ (2). Let $(p_i)_{i\in I}$ be a family of pairwise orthogonal projections in M. Consider the increasing net $x_J = \sum_{i\in J} p_i$, where $J \subset I$ is a finite subset. We have $\sup_J x_J = \sum_{i\in I} p_i$ and so

$$\varphi\left(\sum_{i\in I} p_i\right) = \varphi\left(\sup_J x_J\right) = \sup_J \varphi(x_J) = \sup_J \sum_{i\in J} \varphi(p_i) = \sum_{i\in I} \varphi(p_i).$$

(2) ⇒ (3). Fix $q \in M$ a nonzero projection and $\xi \in \operatorname{ran}(q)$ such that $\varphi(q) \leq 1 < \langle q\xi, \xi\rangle$. There exists a nonzero projection $p \leq q$ such that $\varphi(pxp) \leq \langle pxp\xi, \xi\rangle$ for all $x \in M$. Indeed, by Zorn's Lemma, let $(p_i)_{i\in I}$ be a maximal family of pairwise orthogonal projections in M such that $\varphi(p_i) \geq \langle p_i\xi, \xi\rangle$ for all $i \in I$. By assumption, we have

$$\varphi\left(\sum_{i\in I} p_i\right) = \sum_{i\in I} \varphi(p_i) \geq \sum_{i\in I} \langle p_i\xi, \xi\rangle = \left\langle \left(\sum_{i\in I} p_i\right)\xi, \xi \right\rangle.$$

Put $p = q - \sum_{i\in I} p_i$ and observe that $p \neq 0$. By maximality of the family $(p_i)_{i\in I}$, we have $\varphi(r) < \langle r\xi, \xi\rangle$ for every nonzero projection $r \leq p$. Therefore, using the Spectral Theorem and since φ is $\|\cdot\|_\infty$-continuous, we get $\varphi(pxp) \leq \langle pxp\xi, \xi\rangle$ for all $x \in M_+$. By Cauchy–Schwarz Inequality, we have for all $x \in (M)_1$,

$$|\varphi(xp)|^2 = |\varphi(1^* xp)|^2 \leq \varphi(px^*xp)\varphi(1) \leq \langle px^*xp\xi, \xi\rangle = \|xp\xi\|^2.$$

It follows that $\varphi(\,\cdot\, p)$ is strongly continuous on $(M)_1$.

By Zorn's Lemma, let $(p_i)_{i\in I}$ be a maximal family of pairwise orthogonal projections such that $\varphi(\,\cdot\, p_i)$ is strongly continuous on $(M)_1$ for all $i \in I$. By maximality of the family and the previous reasoning, we have $\sum_{i\in I} p_i = 1$. Therefore $\sum_{i\in I} \varphi(p_i) = \varphi(1) = 1$. Let $\varepsilon > 0$. There exists a finite subset $F \subset I$ such for all finite subsets $F \subset J \subset I$, we have $\varphi(p_J^\perp) = 1 - \varphi(p_J) \leq \varepsilon$, where $p_J = \sum_{i\in J} p_i$. Moreover the Cauchy–Schwarz Inequality yields $|\varphi(xp_J^\perp)|^2 \leq \varphi(p_J^\perp)\varphi(xx^*) \leq \varepsilon$ for all $x \in (M)_1$ and all $F \subset J \subset I$. We have $\|\varphi - \varphi(\,\cdot\, p_J)\| \leq \sqrt{\varepsilon}$ for all $F \subset J \subset I$. Since the net $(\varphi(\,\cdot\, p_J))_J$ converges to φ in M^* and since $\varphi(\,\cdot\, p_J) \in M_*$ for all finite subsets $J \subset I$, we have $\varphi \in M_*$. (3) $\Rightarrow$ (1) is trivial. □

Lemma 2.12. *Let $M \subset \mathbf{B}(H)$ be a von Neumann algebra. Any $\varphi \in M_*$ is a linear combination of four elements in $(M_*)_+$.*

Proof. By Proposition 1.34, there exist $(\xi_n), (\eta_n) \in \ell^2(\mathbf{N}, H)$ such that $\varphi(x) = \sum_n \langle x\xi_n, \eta_n\rangle$. A simple calculation shows that we have

$$\langle x\xi_n, \eta_n\rangle = \frac{1}{4}\sum_{k=0}^{3} \mathrm{i}^k \langle x(\xi_n + \mathrm{i}^k\eta_n), \xi_n + \mathrm{i}^k\eta_n\rangle.$$

It follows that for all $x \in M$, we have

$$\varphi(x) = \frac{1}{4}\sum_{k=0}^{3} \mathrm{i}^k \sum_n \langle x(\xi_n + \mathrm{i}^k\eta_n), \xi_n + \mathrm{i}^k\eta_n\rangle.$$

□

Theorem 2.13. *Any $*$-isomorphism between von Neumann algebras is normal and ultraweakly continuous.*

Proof. Let $\pi : M \to N$ be a $*$-isomorphism. Let (x_i) be a uniformly bounded net of selfadjoint operators in M and write $x = \sup x_i$. We have $\pi(x_i) \leq \pi(x)$ so that $\sup \pi(x_i) \leq \pi(x)$. Write $y = \sup \pi(x_i)$. We have $x_i = \pi^{-1}(\pi(x_i)) \leq \pi^{-1}(y)$ so that $x \leq \pi^{-1}(y)$. Thus $y = \pi(x)$ and π is normal.

For all $\varphi \in (N_*)_+$, $\varphi \circ \pi$ is normal and thus ultraweakly continuous by Theorem 2.11. By Lemma 2.12, we have $\varphi \circ \pi \in M_*$, for all $\varphi \in N_*$. Therefore π is ultraweakly continuous. □

2.3. **Tracial von Neumann algebras.** A von Neumann algebra M is said to be *tracial* if it is endowed with a faithful normal state τ which satisfies the *trace* relation:

$$\tau(xy) = \tau(yx), \forall x, y \in M.$$

Such a tracial state will be refered to as a *trace*. We will say that M is a II_1 factor if M is an infinite dimensional tracial von Neumann algebra and a factor.

Let (M, τ) be a tracial von Neumann algebra. We endow M with the following inner product

$$\langle x, y\rangle_\tau = \tau(y^*x), \forall x, y \in M.$$

Denote by $(\pi_\tau, \mathrm{L}^2(M), \xi_\tau)$ the GNS representation of M with respect to τ. To simplify the notation, we identify $\pi_\tau(x)$ with $x \in M$ and regard $M \subset \mathbf{B}(\mathrm{L}^2(M))$. Define $J : M\xi_\tau \ni x\xi_\tau \mapsto x^*\xi_\tau \in \mathrm{L}^2(M)$. For all $x, y \in M$, we have

$$\langle Jx\xi_\tau, Jy\xi_\tau\rangle = \langle x^*\xi_\tau, y^*\xi_\tau\rangle = \tau(yx^*) = \tau(x^*y) = \langle y\xi_\tau, x\xi_\tau\rangle.$$

Thus $J : \mathrm{L}^2(M) \to \mathrm{L}^2(M)$ is a conjugate linear unitary such that $J^2 = 1$.

Theorem 2.14. *We have $JMJ = M'$.*

Proof. We first prove $JMJ \subset M'$. Let $x, y, a \in M$. We have

$$JxJy\, a\xi_\tau = Jxa^*y^*\xi_\tau = yax^*\xi_\tau = yax^*\xi_\tau = yJxa^*\xi_\tau = yJxJ\, a\xi_\tau$$

so that $JxJy = yJxJ$.

Claim 2.15. The faithful normal state $x \mapsto \langle x\xi_\tau, \xi_\tau\rangle$ is a trace on M'.

Let $x \in M'$. We first show that $Jx\xi_\tau = x^*\xi_\tau$. Indeed, for every $a \in M$, we have

$$\begin{aligned}\langle Jx\xi_\tau, a\xi_\tau\rangle &= \langle Ja\xi_\tau, x\xi_\tau\rangle = \langle x^*a^*\xi_\tau, \xi_\tau\rangle \\ &= \langle a^*x^*\xi_\tau, \xi_\tau\rangle = \langle x^*\xi_\tau, a\xi_\tau\rangle.\end{aligned}$$

Let now $x, y \in M'$. We have

$$\begin{aligned}\langle xy\xi_\tau, \xi_\tau\rangle &= \langle y\xi_\tau, x^*\xi_\tau\rangle = \langle y\xi_\tau, Jx\xi_\tau\rangle = \langle x\xi_\tau, Jy\xi_\tau\rangle \\ &= \langle x\xi_\tau, y^*\xi_\tau\rangle = \langle yx\xi_\tau, \xi_\tau\rangle.\end{aligned}$$

Denote the faithful normal trace $x \mapsto \langle x\xi_\tau, \xi_\tau\rangle$ on M' by τ'. Define the canonical antiunitary K on $\mathrm{L}^2(M', \tau') = \overline{M'\xi_\tau} = \mathrm{L}^2(M)$ by $Kx\xi_\tau = x^*\xi_\tau$, $\forall x \in M'$. The first part of the proof yields $KM'K \subset M'' = M$. Since K and J coincide on $M'\xi_\tau$, which is dense in $\mathrm{L}^2(M)$, it follows that $K = J$. Therefore, we have $JM'J \subset M$ and so $JMJ = M'$. ☐

Definition 2.16. Let $\mathcal{N} \subset \mathcal{M}$ be any inclusion of von Neumann algebras. A *conditional expectation* $\mathrm{E} : \mathcal{M} \to \mathcal{N}$ is a contractive unital $\mathcal{N}$-$\mathcal{N}$-bimodular linear map.

We next show that for inclusions of tracial von Neumann algebras $N \subset M$, there always exists a conditional expectation $\mathrm{E} : M \to N$.

Theorem 2.17. *Let $N \subset M$ be any inclusion of tracial von Neumann algebras and $\tau \in M_*$ a distinguished faithful normal trace. Then there exists a unique trace preserving conditional expectation $\mathrm{E}_N : M \to N$.*

Proof. We still denote by τ the faithful normal trace $\tau|_N \in N_*$. Regard $\mathrm{L}^2(N)$ as a closed subspace of $\mathrm{L}^2(M)$ via the identity mapping $\mathrm{L}^2(N) \to \mathrm{L}^2(M) : x\xi_\tau \mapsto x\xi_\tau$. For all $T \in M$, define a sesquilinear form $\kappa_T : \mathrm{L}^2(N) \times \mathrm{L}^2(N) \to \mathbf{C}$ by the formula

$$\kappa_T(x\xi_\tau, y\xi_\tau) = \tau(y^*Tx).$$

By Cauchy–Schwarz inequality, we have $|\kappa_T(x\xi_\tau, y\xi_\tau)| \leq \|T\|_\infty \|x\|_2 \|y\|_2$ for all $x, y \in N$ and hence there exists $\mathrm{E}_N(T) \in \mathbf{B}(\mathrm{L}^2(N))$ such that $\kappa_T(x\xi_\tau, y\xi_\tau) = \langle \mathrm{E}_N(T)x\xi_\tau, y\xi_\tau\rangle$ for all $x, y \in$

N. Observe that $\|\mathrm{E}_N(T)\|_\infty \leq \|T\|_\infty$. For all $x, y, a \in N$, we have

$$\begin{aligned}\langle \mathrm{E}_N(T)Ja^*J\,x\xi_\tau, y\xi_\tau\rangle &= \langle \mathrm{E}_N(T)xa\xi_\tau, y\xi_\tau\rangle \\ &= \tau(y^*Txa) \\ &= \tau((ya^*)^*Tx) \\ &= \langle \mathrm{E}_N(T)x\xi_\tau, ya^*\xi_\tau\rangle \\ &= \langle \mathrm{E}_N(T)x\xi_\tau, JaJy\xi_\tau\rangle \\ &= \langle Ja^*J\mathrm{E}_N(T)\,x\xi_\tau, y\xi_\tau\rangle.\end{aligned}$$

This implies that $\mathrm{E}(T) \in (JNJ)' = N$. It is routine to check that $\mathrm{E}_N : M \to N$ is a trace preserving conditional expectation.

We next show that there is a unique trace preserving conditional expectation $\mathrm{E} : M \to N$. Indeed, for all $T \in M$ and all $x, y \in N$, we have

$$\begin{aligned}\langle \mathrm{E}(T)x\xi_\tau, y\xi_\tau\rangle &= \tau(y^*\mathrm{E}(T)x) \\ &= \tau(\mathrm{E}(y^*Tx)) \\ &= \tau(y^*Tx) \\ &= \langle \mathrm{E}_N(T)x\xi_\tau, y\xi_\tau\rangle.\end{aligned}$$

This shows that $\mathrm{E}(T) = \mathrm{E}_N(T)$ for every $T \in M$ and hence $\mathrm{E} = \mathrm{E}_N$. □

3. Group von Neumann algebras and group measure space constructions

3.1. **Group von Neumann algebras.** Let Γ be a countable discrete group. The *left* regular representation $\lambda : \Gamma \to \mathcal{U}(\ell^2(\Gamma))$ is defined by $\lambda_s\delta_t = \delta_{st}$ for all $s, t \in \Gamma$.

Definition 3.1 (Group von Neumann algebra)**.** The von Neumann algebra $\mathrm{L}(\Gamma)$ is defined as the weak closure of the linear span of $\{\lambda_s : s \in \Gamma\}$.

Likewise, we can define the *right* regular representation $\rho : \Gamma \to \mathcal{U}(\ell^2(\Gamma))$ by $\rho_s\delta_t = \delta_{ts^{-1}}$ for all $s, t \in \Gamma$. The *right* von Neumann algebra $\mathrm{R}(\Gamma)$ is defined as the weak closure of the linear span of $\{\rho_s : s \in \Gamma\}$. We obviously have $\mathrm{L}(\Gamma) \subset \mathrm{R}(\Gamma)'$.

Proposition 3.2. *The vector state* $\tau : \mathrm{L}(\Gamma) \to \mathbf{C}$ *defined by* $\tau(x) = \langle x\delta_e, \delta_e\rangle$ *is a faithful normal trace. Moreover* $\mathrm{L}(\Gamma) = \mathrm{R}(\Gamma)'$.

Proof. It is clear that τ is normal. We moreover have

$$\tau(\lambda_s\lambda_t) = \tau(\lambda_{st}) = \delta_{st,e} = \delta_{ts,e} = \tau(\lambda_{ts}) = \tau(\lambda_t\lambda_s).$$

It follows that τ is a trace on $\mathrm{L}(\Gamma)$. Assume now that $\tau(x^*x) = 0$, that is, $x\delta_e = 0$ for $x \in \mathrm{L}(\Gamma)$. For all $t \in \Gamma$, we have $x\delta_t = x\rho_{t^{-1}}\delta_e = \rho_{t^{-1}}x\delta_e = 0$. Therefore $x = 0$. Hence τ is faithful.

We can identify $\ell^2(\Gamma)$ with $\mathrm{L}^2(\mathrm{L}(\Gamma))$ via the unitary mapping $\delta_g \mapsto u_g$. Under this identification, we have $J\delta_t = \delta_{t^{-1}}$. An easy calculation shows that for all $s, t \in \Gamma$, we have

$$J\lambda_sJ\,\delta_t = J\lambda_s\delta_{t^{-1}} = J\delta_{st^{-1}} = \delta_{ts^{-1}} = \rho_s\,\delta_t.$$

Therefore, $J\lambda_sJ = \rho_s$ for all $s \in \Gamma$. It follows that $\mathrm{L}(\Gamma)' = J\mathrm{L}(\Gamma)J = \mathrm{R}(\Gamma)$ and thus $\mathrm{L}(\Gamma) = \mathrm{R}(\Gamma)'$. □

Let $x \in \mathrm{L}(\Gamma)$ and write $x\delta_e = \sum_{s\in\Gamma} x_s\delta_s \in \ell^2(\Gamma)$ with $x_s = \langle x\delta_e, \delta_s\rangle = \tau(x\lambda_s^*)$ for all $s \in \Gamma$. As we have seen, the family $(x_s)_{s\in\Gamma}$ completely determines $x \in \Gamma$. We shall denote by $x = \sum_{s\in\Gamma} x_s\lambda_s$ the *Fourier expansion* of $x \in \mathrm{L}(\Gamma)$.

The above sum $\sum_{s\in\Gamma} x_s\lambda_s$ **does not converge** in general for any of the topologies on $\mathbf{B}(\ell^2(\Gamma))$. However, the net of finite sums $(x_{\mathcal{F}})_{\mathcal{F}}$ defined by $x_{\mathcal{F}} = \sum_{s\in\mathcal{F}} x_s\lambda_s$ for $\mathcal{F} \subset \Gamma$ a finite subset does converge for the $\|\cdot\|_2$-norm. Indeed since $(x_s) \in \ell^2(\Gamma)$, for any $\varepsilon > 0$, there exists $\mathcal{F}_0 \subset \Gamma$ finite subset such that $\sum_{s\in\Gamma\setminus\mathcal{F}_0} |x_s|^2 \leq \varepsilon^2$. Thus, for every finite subset $\mathcal{F} \subset \Gamma$ such that $\mathcal{F}_0 \subset \mathcal{F}$, we have $\|x - x_{\mathcal{F}}\|_2^2 = \sum_{s\in\Gamma\setminus\mathcal{F}} |x_s|^2 \leq \varepsilon^2$.

The notation $x = \sum_{s\in\Gamma} x_s\lambda_s$ behaves well with respect to taking the adjoint and multiplication.

Proposition 3.3. *Let $x = \sum_{s\in\Gamma} x_s\lambda_s$ (resp. $y = \sum_{t\in\Gamma} y_t\lambda_t$) be the Fourier expansion of $x \in \mathrm{L}(\Gamma)$ (resp. $y \in \mathrm{L}(\Gamma)$). Then we have*

- $x^* = \sum_{s\in\Gamma} \overline{x_{s^{-1}}}\lambda_s$.
- $xy = \sum_{t\in\Gamma} \left(\sum_{t\in\Gamma} x_s y_{s^{-1}t}\right)\lambda_t$, *with* $\sum_{s\in\Gamma} x_s y_{s^{-1}t} \in \mathbf{C}$ *for all* $t \in \Gamma$, *by Cauchy–Schwarz inequality.*

Proof. For the first item, observe that

$$(x^*)_s = \tau(x^*\lambda_s^*) = \overline{\tau(\lambda_s x)} = \overline{\tau(x\lambda_{s^{-1}}^*)} = \overline{x_{s^{-1}}}.$$

For the second item, observe that using Cauchy–Schwarz inequality, we have

$$(xy)_t = \tau(xy\lambda_t^*) = \sum_{s\in\Gamma} x_s\tau(\lambda_s y\lambda_t^*) = \sum_{s\in\Gamma} x_s\tau(y\lambda_{s^{-1}t}^*) = \sum_{s\in\Gamma} x_s y_{s^{-1}t}. \qquad \square$$

Thanks to the Fourier expansion, we can compute the center $\mathcal{Z}(\mathrm{L}(\Gamma))$ of the group von Neumann algebra. We say that Γ is icc (infinite conjugacy classes) if for every $s \in \Gamma \setminus \{e\}$, the conjugacy class $\{tst^{-1} : t \in \Gamma\}$ is infinite.

Proposition 3.4. *We have $x = \sum_{s\in\Gamma} x_s\lambda_s \in \mathcal{Z}(\mathrm{L}(\Gamma))$ if and only if $x_{tst^{-1}} = x_s$ for all $s, t \in \Gamma$. In particular, $\mathrm{L}(\Gamma)$ is a factor if and only if Γ is icc. Thus, $\mathrm{L}(\Gamma)$ is a II_1 factor whenever Γ is infinite and icc.*

Proof. We have

$$\begin{aligned} x = \sum_{s\in\Gamma} x_s\lambda_s \in \mathcal{Z}(\mathrm{L}(\Gamma)) &\Leftrightarrow \lambda_t^* x\lambda_t = x, \forall s \in \Gamma \\ &\Leftrightarrow x_{tst^{-1}} = x_s, \forall s, t \in \Gamma. \end{aligned}$$

If Γ is icc and $x \in \mathcal{Z}(\mathrm{L}(\Gamma))$, since $(x_{tst^{-1}})_t \in \ell^2(\Gamma)$, for all $s \in \Gamma$, it follows that $x_s = 0$ for all $s \in \Gamma \setminus \{e\}$. Hence $\mathcal{Z}(\mathrm{L}(\Gamma)) = \mathbf{C}$.

If Γ is not icc, then $F = \{tst^{-1} : t \in \Gamma\}$ is finite for some $s \in \Gamma \setminus \{e\}$. Then $\sum_{h\in F} \lambda_h \in \mathcal{Z}(\mathrm{L}(\Gamma)) \setminus \mathbf{C}$. $\square$

Example 3.5. Here are a few examples of icc groups: the subgroup $S_\infty < S(\mathbf{N})$ of finitely supported permutations; the free groups $\mathbf{F}_n$ for $n \geq 2$; the lattices $\mathrm{PSL}(n, \mathbf{Z})$ for $n \geq 2$.

Hence Proposition 3.4 provides many examples of II_1 factors arising from countable discrete groups.

Exercise 3.6. Let $T = [T_{st}]_{s,t\in\Gamma} \in \mathbf{B}(\ell^2(\Gamma))$, with $T_{st} = \langle T\delta_t, \delta_s\rangle$. Show that $T \in \mathrm{L}(\Gamma)$ if and only if T is *constant down the diagonals*, that is, $T_{st} = T_{gh}$ whenever $st^{-1} = gh^{-1}$.

Example 3.7. Assume that Γ is abelian. Then the Pontryagin dual $\widehat{\Gamma}$ is a compact second countable abelian group. Write $\mathcal{F} : \ell^2(\Gamma) \to \mathrm{L}^2(\widehat{\Gamma}, \mathrm{Haar})$ for the Fourier transform which is defined by $\mathcal{F}(\delta_s)(\chi) = \langle s, \chi\rangle$. Observe that $\mathcal{F}$ is a unitary operator. We then get

$$\mathrm{L}^\infty(\widehat{\Gamma}) = \mathcal{F}\mathrm{L}(\Gamma)\mathcal{F}^*.$$

3.2. **Murray–von Neumann's group measure space construction.** Let $\Gamma \curvearrowright (X, \mu)$ be a probability measure preserving (pmp) action. Define the action $\sigma : \Gamma \curvearrowright \mathrm{L}^\infty(X)$ by $(\sigma_s(F))(x) = F(s^{-1}x)$, $\forall F \in \mathrm{L}^\infty(X)$. This action extends to a unitary representation $\sigma : \Gamma \to \mathcal{U}(\mathrm{L}^2(X))$. Put $H = \mathrm{L}^2(X) \otimes \ell^2(\Gamma)$. Put $u_s = \sigma_s \otimes \lambda_s$ for all $s \in \Gamma$. Observe that by Fell's absorption principle, the representation $\Gamma \to \mathcal{U}(H) : s \mapsto u_s$ is unitarily conjugate to a multiple of the left regular representation. We will identify $F \in \mathrm{L}^\infty(X)$ with $F \otimes 1 \in \mathrm{L}^\infty(X) \otimes 1$.

We have the following *covariance* relation:

$$u_s F u_s^* = \sigma_s(F), \forall F \in \mathrm{L}^\infty(X), \forall s \in \Gamma.$$

Definition 3.8 (Murray, von Neumann [MvN43])**.** The *group measure space construction* $\mathrm{L}^\infty(X) \rtimes \Gamma$ is defined as the weak closure of the linear span of $\{F u_s : F \in \mathrm{L}^\infty(X), s \in \Gamma\}$.

Put $M = \mathrm{L}^\infty(X) \rtimes \Gamma$. Define the unital faithful $*$-representation $\pi : \mathrm{L}^\infty(X) \to \mathbf{B}(H)$ by $\pi(F)(\xi \otimes \delta_t) = \sigma_t(F)\xi \otimes \delta_t$. Denote by N the von Neumann algebra acting on H generated by $\pi(\mathrm{L}^\infty(X))$ and $(1 \otimes \rho)(\Gamma)$. It is straightforward to check that $M \subset N'$.

Proposition 3.9. *The vector state $\tau : M \to \mathbf{C}$ defined by $\tau(x) = \langle x(\mathbf{1}_X \otimes \delta_e), \mathbf{1}_X \otimes \delta_e\rangle$ is a faithful normal trace. Moreover we have $M = N'$.*

Proof. It is clear that τ is normal. We moreover have

$$\begin{aligned}
\tau(F u_s\, G u_t) &= \tau(F\sigma_s(G)u_{st}) \\
&= \delta_{st,e} \int_X F(x)G(s^{-1}x)\,\mathrm{d}\mu(x) \\
&= \delta_{st,e} \int_X F(sx)G(x)\,\mathrm{d}\mu(x) \\
&= \delta_{ts,e} \int_X G(x)F(t^{-1}x)\,\mathrm{d}\mu(x) \\
&= \tau(G\sigma_t(F)u_{ts}) \\
&= \tau(G u_t\, F u_s).
\end{aligned}$$

It follows that τ is a trace on M. Assume that $\tau(b^*b) = 0$, that is, $b(\mathbf{1}_X \otimes \delta_e) = 0$. For all $s \in \Gamma$ and all $F \in \mathrm{L}^\infty(X)$, we have

$$\begin{aligned}
b\,(F \otimes \delta_t) &= b\,\pi(\sigma_{t^{-1}}(F))(1 \otimes \rho_{t^{-1}})(\mathbf{1}_X \otimes \delta_e) \\
&= \pi(\sigma_{t^{-1}}(F))(1 \otimes \rho_{t^{-1}})\, b(\mathbf{1}_X \otimes \delta_e) = 0.
\end{aligned}$$

It follows that $b = 0$. Hence τ is faithful.

We will identify $\mathrm{L}^2(M)$ with $\mathrm{L}^2(X)\otimes\ell^2(\Gamma)$ via the unitary mapping $F u_s \xi_\tau \mapsto F\otimes\delta_s$. Under this identification, the conjugation $J : \mathrm{L}^2(M) \to \mathrm{L}^2(M)$ is defined by $J(\xi \otimes \delta_s) = \sigma_{s^{-1}}(\xi^*) \otimes \delta_{s^{-1}}$. For all $F \in \mathrm{L}^\infty(X)$ and all $s \in \Gamma$, we have

$$\begin{aligned}
J(\sigma_s \otimes \lambda_s)J &= 1 \otimes \rho_s \\
J(F \otimes 1)J &= \pi(F)^*.
\end{aligned}$$

Therefore, we get $M = N'$. ∎

Observe that when the probability space $X = \{\bullet\}$ is a point, then the group von Neumann algebra and the group measure space construction coincide, that is, $\mathrm{L}^\infty(X) \rtimes \Gamma = \mathrm{L}(\Gamma)$.

Proposition 3.10 (Fourier expansion). *Let $\Gamma \curvearrowright (X, \mu)$ be a pmp action. Let $A = \mathrm{L}^\infty(X)$ and $M = \mathrm{L}^\infty(X) \rtimes \Gamma$. Denote by $\mathrm{E}_A : M \to A$ the unique trace preserving conditional expectation. Every $a \in M$ has a unique* Fourier expansion *of the form $a = \sum_{s\in\Gamma} a_s u_s$ with $a_s = \mathrm{E}_A(au_s^*)$. The convergence holds for the $\|\cdot\|_2$-norm. Moreover, we have the following:*

- $a^* = \sum_{s\in\Gamma} \sigma_{s^{-1}}(a_s^*) u_s$.
- $\|a\|_2^2 = \sum_{s\in\Gamma} \|a_s\|_2^2$.
- $ab = \sum_{t\in\Gamma} \left(\sum_{s\in\Gamma} a_s \sigma_s(b_{s^{-1}t})\right) u_t$.

Proof. Define the unitary mapping $U : \mathrm{L}^2(M) \to \mathrm{L}^2(X) \otimes \ell^2(\Gamma)$ by the formula $U(au_s\xi_\tau) = a \otimes \delta_s$. Then $U\xi_\tau = \mathbf{1}_X \otimes \delta_e$ is a cyclic separating vector for M represented on the Hilbert space $\mathrm{L}^2(X) \otimes \ell^2(\Gamma)$. We identify $\mathrm{L}^2(M)$ with $\mathrm{L}^2(X) \otimes \ell^2(\Gamma)$. Under this identification, e_A is the orthogonal projection $\mathrm{L}^2(X) \otimes \ell^2(\Gamma) \to \mathrm{L}^2(X) \otimes \mathbf{C}\delta_e$. Moreover, $u_s e_A u_s^*$ is the orthogonal projection $\mathrm{L}^2(X) \otimes \ell^2(\Gamma) \to \mathrm{L}^2(X) \otimes \mathbf{C}\delta_s$ and thus $\sum_{s\in\Gamma} u_s e_A u_s^* = 1$. Let $a \in M$. Regarding $a(\mathbf{1}_X \otimes \delta_e) \in \mathrm{L}^2(X) \otimes \ell^2(\Gamma)$, we know that there exists $a_s \in \mathrm{L}^2(X)$ such that

$$a(\mathbf{1}_X \otimes \delta_e) = \sum_{s\in\Gamma} a_s \otimes \delta_s \quad \text{and} \quad \|a\|_2^2 = \sum_{s\in\Gamma} \|a_s\|_2^2.$$

Then we have

$$\begin{aligned} a_s \otimes \delta_s &= u_s e_A u_s^* a(\mathbf{1}_X \otimes \delta_e) \\ &= u_s e_A u_s^* a e_A(\mathbf{1}_X \otimes \delta_e) \\ &= u_s \mathrm{E}_A(u_s^* a)(\mathbf{1}_X \otimes \delta_e) \\ &= \mathrm{E}_A(au_s^*) \otimes \delta_s. \end{aligned}$$

It follows that $a_s = \mathrm{E}_A(au_s^*)$. Therefore, we have $a = \sum_{s\in\Gamma} \mathrm{E}_A(au_s^*) u_s$ and the convergence holds for the $\|\cdot\|_2$-norm. Moreover, $\|a\|_2^2 = \sum_{s\in\Gamma} \|\mathrm{E}_A(au_s^*)\|_2^2$. The rest of the proof is left to the reader. ☐

Like in the group case, the sum $a = \sum_{s\in\Gamma} a_s u_s$ **does not converge** in general for any of the operator topologies on $\mathbf{B}(\mathrm{L}^2(X) \otimes \ell^2(\Gamma))$.

Definition 3.11. Let $\Gamma \curvearrowright (X, \mu)$ be a pmp action.

- We say that the action is *(essentially) free* if $\mu(\{x \in X : sx = x\}) = 0$ for all $s \in \Gamma \setminus \{e\}$.
- We say that the action is *ergodic* if every Γ-invariant measurable subset $U \subset X$ has measure 0 or 1.

Lemma 3.12. *Let $\Gamma \curvearrowright (X, \mu)$ be a pmp action and denote by $\sigma : \Gamma \to \mathrm{L}^2(X)^0$ the corresponding Koopman representation where $\mathrm{L}^2(X)^0 = \mathrm{L}^2(X) \ominus \mathbf{C}\mathbf{1}_X$. The following are equivalent:*

(1) The action $\Gamma \curvearrowright (X, \mu)$ is ergodic.
(2) The Koopman representation $\sigma \to \mathcal{U}(\mathrm{L}^2(X)^0)$ has no nonzero invariant vectors.

Proof. (1) $\Rightarrow$ (2) Let $\xi \in \mathrm{L}^2(X)^0$ such that $\sigma_s(\xi) = \xi$ for all $s \in \Gamma$. By considering the real part and the imaginary part of $\xi \in \mathrm{L}^2(X)^0$, we may further assume that $\xi \in \mathrm{L}^2(X)^0$ is real-valued. For every $t \in \mathbf{R}$, define $U_t = \{x \in X : \xi(x) \geq t\}$. It follows that U_t is Γ-invariant for all $t \in \mathbf{R}$ and thus $\mu(U_t) = 0, 1$ by ergodicity. Since the fonction $t \mapsto \mu(U_t)$ is decreasing and since

$\xi \in \mathrm{L}^2(X)$, there exists $t_0 \in \mathbf{R}$ such that $\mu(U_t) = 1$ for all $t < t_0$ and $\mu(U_t) = 0$ for all $t > t_0$. Therefore $\xi(x) = t_0$ for μ-almost every $x \in X$. Since $\xi \in \mathrm{L}^2(X)^0$, we get $t_0 = 0$ and so $\xi = 0$.

(2) $\Rightarrow$ (1) Let $U \subset X$ be a Γ-invariant measurable subset. Put $\xi = \mathbf{1}_U - \mu(U)\mathbf{1}_X \in \mathrm{L}^2(X)^0$. Since $\sigma_s(\xi) = \xi$ for all $s \in \Gamma$, we get $\xi = 0$ and so $\mathbf{1}_U = \mu(U)\mathbf{1}_X$. Hence $\mu(U) = 0, 1$. □

Examples 3.13. Here are a few examples of pmp free ergodic actions $\Gamma \curvearrowright (X, \mu)$.

(1) **Bernoulli actions.** Let Γ be an infinite group and (Y, η) a nontrivial probability space, that is, η is not a Dirac point mass. Put $(X, \mu) = (Y^\Gamma, \nu^{\otimes \Gamma})$. Consider the Bernoulli action $\Gamma \curvearrowright Y^\Gamma$ defined by

$$s \cdot (y_t)_{t \in \Gamma} = (y_{s^{-1}t})_{t \in \Gamma}.$$

Then the Bernoulli action is pmp free and mixing, so in particular ergodic.

(2) **Profinite actions.** Let Γ be an infinite residually finite group together with a decreasing chain of finite index normal subgroups $\Gamma_n \lhd \Gamma$ such that $\Gamma_0 = \Gamma$ and $\cap_{n \in \mathbf{N}} \Gamma_n = \{e\}$. Then for all $n \geq 1$, the action $\Gamma \curvearrowright (\Gamma/\Gamma_n, \mu_n)$ is transitive and preserves the normalized counting measure μ_n. Consider the profinite action defined as the projective limit

$$\Gamma \curvearrowright (\mathbf{G}, \mu) = \varprojlim \Gamma \curvearrowright (\Gamma/\Gamma_n, \mu_n).$$

Then Γ sits as a dense subgroup of the compact group $\mathbf{G}$ which is the profinite completion of Γ with respect to the decreasing chain $(\Gamma_n)_{n \in \mathbf{N}}$. Observe that μ is the unique Haar probability measure on $\mathbf{G}$. The profinite action is pmp free and ergodic.

(3) **Actions on tori.** Let $n \geq 2$. Consider the action $\mathrm{SL}(n, \mathbf{Z}) \curvearrowright (\mathbf{T}^n, \lambda_n)$ where $\mathbf{T}^n = \mathbf{R}^n/\mathbf{Z}^n$ is the n-torus and λ_n is the unique Haar probability measure. This action is pmp free and ergodic.

We always assume that (X, μ) is a standard probability space. In particular, X is *countably separated* in the sense that there exists a sequence of Borel subsets $V_n \subset X$ such that $\bigcup_n V_n = X$, $\mu(V_n) > 0$ for all $n \in \mathbf{N}$ and with the property that whenever $x, y \in X$ and $x \neq y$, there exists $n \in \mathbf{N}$ for which $x \in V_n$ and $y \notin V_n$.

Proposition 3.14. *Let $\Gamma \curvearrowright (X, \mu)$ be a pmp action. Put $A = \mathrm{L}^\infty(X)$ and $M = \mathrm{L}^\infty(X) \rtimes \Gamma$.*

(1) The action is free if and only if $A \subset M$ is maximal abelian, that is, $A' \cap M = A$.
(2) Under the assumption that the action is free, the action is ergodic if and only if M is a factor.

Proof. (1) Assume that the action is free. Let $b \in A' \cap M$ and write $b = \sum_{s \in \Gamma} b_s u_s$ for its Fourier expansion. Then for all $a \in A$ and all $s \in \Gamma$, we have $ab_s = \sigma_s(a)b_s$. Fix $s \in \Gamma \setminus \{e\}$ and put $U_s = \{x \in X : b_s(x) \neq 0, sx \neq x\}$. We have $\mathbf{1}_{U_s} a = \mathbf{1}_{U_s}\sigma_s(a)$ for all $a \in A$.

By assumption, we have $U_s = U_s \cap (\bigcup_n V_n \cap s(V_n)^c)$. So, if $\mu(U_s) > 0$, there exists $n \in \mathbf{N}$ such that $\mu(U_s \cap V_n \cap s(V_n)^c) > 0$. With $a = \mathbf{1}_{V_n}$, we get $\mathbf{1}_{U_s \cap V_n} = \mathbf{1}_{U_s \cap s(V_n)}$ and thus $\mathbf{1}_{U_s \cap V_n \cap s(V_n)^c} = 0$, which is a contradiction. Therefore, $\mu(U_s) = 0$. Since the action is moreover free, we get $b_s = 0$. This implies that $b \in A$.

Conversely, assume that $A' \cap M = A$. For all $s \in \Gamma \setminus \{e\}$, put $a_s = \mathbf{1}_{\{x \in X : sx = x\}}$. We have $a_s u_s \in A' \cap M = A$. Hence $a_s u_s = \mathrm{E}_A(a_s u_s) = 0$ and so $a_s = 0$. Therefore $\mu(\{x \in X : sx = x\}) = 0$.

(2) Under the assumption that the action is free, we have $\mathcal{Z}(M) = M' \cap M = M' \cap A = A^\Gamma$. Therefore, the action is ergodic if and only if $\mathcal{Z}(M) = \mathbf{C}$. □

Let $A \subset M$ be any inclusion of von Neumann algebras. Denote by $\mathcal{N}_M(A) := \{u \in \mathcal{U}(M) : uAu^* = A\}$ the group of unitaries normalizing A inside M and by $\mathcal{N}_M(A)''$ the *normalizer* of A inside M. We say that $A \subset M$ is a *Cartan subalgebra* when the following three conditions are satisfied:

(1) A is maximal abelian, that is, $A = A' \cap M$;
(2) There exists a faithful normal conditional expectation $\mathrm{E}_A : M \to A$;
(3) $\mathcal{N}_M(A)'' = A$.

For every free pmp action $\Gamma \curvearrowright (X, \mu)$, $\mathrm{L}^\infty(X) \subset \mathrm{L}^\infty(X) \rtimes \Gamma$ is a Cartan subalgebra by Proposition 3.14.

4. Amenable von Neumann algebras

4.1. **Connes's theory of bimodules.** The discovery of the appropriate notion of representations for von Neumann algebras, as so-called *correspondences* or *bimodules*, is due to Connes. Whenever M is a von Neumann algebra, we denote by M^{op} its opposite von Neumann algebra.

Definition 4.1. Let M, N be tracial von Neumann algebras. A Hilbert space $\mathcal{H}$ is said to be an M-N-*bimodule* if it comes equipped with two commuting normal unital $*$-representations $\lambda : M \to \mathbf{B}(\mathcal{H})$ and $\rho : N^{\mathrm{op}} \to \mathbf{B}(\mathcal{H})$. We shall intuitively write

$$x\xi y = \lambda(x)\rho(y^{\mathrm{op}})\xi, \ \forall \xi \in \mathcal{H}, \forall x \in M, \forall y \in N.$$

We will sometimes denote by $\pi_{\mathcal{H}} : M \otimes_{\mathrm{alg}} N^{\mathrm{op}} \to \mathbf{B}(\mathcal{H})$ the unital $*$-representation associated with the M-N-bimodule structure on $\mathcal{H}$.

Examples 4.2. The following are important examples of bimodules:

(1) The identity M-M-bimodule $\mathrm{L}^2(M)$ with $x\xi y = x\, Jy^*J\xi$.
(2) The coarse M-N-bimodule $\mathrm{L}^2(M) \otimes \mathrm{L}^2(N)$ with $x(\xi \otimes \eta)y = (x\xi) \otimes (\eta y)$.
(3) For any τ-preserving automorphism $\theta \in \mathrm{Aut}(M)$, we regard $\mathrm{L}^2_\theta(M)$ with the following M-M-bimodule structure: $x\xi y = x\xi\theta(y)$.

We will say that two M-N-bimodules ${}_M\mathcal{H}_N$ and ${}_M\mathcal{K}_N$ are *isomorphic* and write ${}_M\mathcal{H}_N \cong {}_M\mathcal{K}_N$ if there exists a unitary mapping $U : \mathcal{H} \to \mathcal{K}$ such that

$$U(x\xi y) = xU(\xi)y, \forall \xi \in \mathcal{H}, \forall x \in M, \forall y \in N.$$

We now describe *Connes's fusion tensor product* for Hilbert bimodules. Let M, N, P be any tracial von Neumann algebras, $\mathcal{H}$ any M-N-bimodule and $\mathcal{K}$ any N-P-bimodule. Denote by $\mathcal{H}_0 \subset \mathcal{H}$ the subspace of right N-bounded vectors, that is,

$$\mathcal{H}_0 := \{a \in \mathcal{H} : \exists c > 0, \forall y \in N, \|ay\| \leq c\|y\|_2\}.$$

Whenever $a \in \mathcal{H}_0$, we denote by $L_a : \mathrm{L}^2(N) \to \mathcal{H} : y\xi_\tau \mapsto ay$ the corresponding bounded operator. Observe that for all $a, b \in \mathcal{H}_0$, we have

$$L_b^* L_a \in (JNJ)' \cap \mathbf{B}(\mathrm{L}^2(N)) = N.$$

Observe that $\mathcal{H}_0$ is dense in $\mathcal{H}$. Indeed, for every $\xi \in \mathcal{H}$, denote by $T_\xi \in \mathrm{L}^1(N, \tau)$ the unique element such that $\langle \xi y, \xi \rangle = \tau(T_\xi y)$ for all $y \in N$. Regarding T_ξ as a closed summable operator affiliated with N, we may take the spectral decomposition of T_ξ and find an increasing sequence of projection $e_n \in N$ such that $\xi e_n \in \mathcal{H}_0$ and $\xi e_n \to \xi$.

The separation/completion of $\mathcal{H}_0 \otimes_{\rm alg} \mathcal{K}$ with respect to the sesquilinear form

$$\langle a \otimes \xi, b \otimes \eta \rangle = \langle L_b^* L_a \xi, \eta \rangle_{\mathcal{K}}$$

is denoted by $\mathcal{H} \otimes_N \mathcal{K}$. The image of $a \otimes \eta \in \mathcal{H}_0 \otimes_{\rm alg} \mathcal{K}$ in $\mathcal{H} \otimes_N \mathcal{K}$ is simply denoted by $a \otimes_N \xi$. The M-P-bimodule structure on $\mathcal{H} \otimes_N \mathcal{K}$ is given by

$$x(a \otimes_N \xi)y = xa \otimes_N \xi y, \forall x \in M, \forall y \in P.$$

Exercise 4.3 (Associativity)**.** Let M, N, P, Q be any tracial von Neumann algebras and ${}_M\mathcal{K}_N$, ${}_N\mathcal{K}_P$, ${}_P\mathcal{L}_Q$ bimodules. Show that as M-Q-bimodules, we have

$${}_M((\mathcal{H} \otimes_N \mathcal{K}) \otimes_P \mathcal{L})_Q \cong {}_M(\mathcal{H} \otimes_N (\mathcal{K} \otimes_P \mathcal{L}))_Q.$$

Like for unitary group representations, we can define a notion of *weak containment* of Hilbert bimodules. Let M, N be any tracial von Neumann algebras and ${}_M\mathcal{H}_N, {}_M\mathcal{K}_N$ any bimodules. Consider the unital $*$-representations $\pi_{\mathcal{H}} : M \otimes_{\rm alg} N^{\rm op} \to \mathbf{B}(\mathcal{H})$ and $\pi_{\mathcal{K}} : M \otimes_{\rm alg} N^{\rm op} \to \mathbf{B}(\mathcal{K})$.

Definition 4.4 (Weak containment)**.** We say that $\mathcal{H}$ is *weakly contained* in $\mathcal{K}$ and write $\mathcal{H} \subset_{\rm weak} \mathcal{K}$ if $\|\pi_{\mathcal{H}}(T)\| \leq \|\pi_{\mathcal{K}}(T)\|$ for all $T \in M \otimes_{\rm alg} N^{\rm op}$.

Let $\pi : \Gamma \to \mathcal{U}(K_\pi)$ be a unitary representation of a countable discrete group Γ. Put $M = \mathrm{L}(\Gamma)$ and denote by $(\lambda_s)_{s \in \Gamma}$ the canonical unitaries in M. Define on $\mathcal{H}(\pi) = K_\pi \otimes \ell^2(\Gamma)$ the following M-M-bimodule structure. For all $\xi \in K_\pi$ and all $s, t \in \Gamma$, define

$$\begin{aligned} \lambda_s \, (\xi \otimes \delta_t) &= \pi_s(\xi) \otimes \delta_{st} \\ (\xi \otimes \delta_t) \, \lambda_s &= \xi \otimes \delta_{ts}. \end{aligned}$$

It is clear that the right multiplication extends to the whole von Neumann algebra M. Observe now that the unitary representations $\pi \otimes \lambda$ and $1_{K_\pi} \otimes \lambda$ are unitarily conjugate. Indeed, define $U : K_\pi \otimes \ell^2(\Gamma) \to K_\pi \otimes \ell^2(\Gamma)$ by

$$U(\xi \otimes \delta_t) = \pi_t(\xi) \otimes \delta_t.$$

It is routine to check that U is a unitary and $U(1_{K_\pi} \otimes \lambda_s)U^* = \pi_s \otimes \lambda_s$ for every $s \in \Gamma$. Therefore, the left multiplication extends to M. Denote by $1_\Gamma : \Gamma \to \mathcal{U}(\mathbf{C})$ the trivial representation.

Proposition 4.5 (Representations and Bimodules)**.** *The formulae above endow the Hilbert space $\mathcal{H}(\pi) = K_\pi \otimes \ell^2(\Gamma)$ with a structure of M-M-bimodule. Moreover, the following assertions hold true:*

(1) ${}_M\mathcal{H}(1_\Gamma)_M \cong {}_M\mathrm{L}^2(M)_M$ *and* ${}_M\mathcal{H}(\lambda_\Gamma)_M \cong {}_M(\mathrm{L}^2(M) \otimes \mathrm{L}^2(M))_M$.

(2) For all unitary Γ-representations π_1 and π_2 such that $\pi_1 \subset_{\rm weak} \pi_2$, we have

$${}_M\mathcal{H}(\pi_1)_M \subset_{\rm weak} {}_M\mathcal{H}(\pi_2)_M.$$

(3) Whenever π_1 and π_2 are unitary Γ-representations, we have

$${}_M\mathcal{H}(\pi_1 \otimes \pi_2)_M \cong {}_M(\mathcal{H}(\pi_1) \otimes_M \mathcal{H}(\pi_2))_M.$$

Proof. The proof is left as an exercise. ∎

4.2. **Powers–Størmer's inequality.** For an inclusion of von Neumann algebra $M \subset \mathcal{N}$, we say that a state $\varphi \in \mathcal{N}^*$ is M-central if $\varphi(xT) = \varphi(Tx)$ for all $x \in M$ and all $T \in \mathcal{N}$. We will be using the following notation: for all $x \in M$, put $\overline{x} = (x^{\mathrm{op}})^* \in M^{\mathrm{op}}$.

Regarding $M \otimes_{\mathrm{alg}} M^{\mathrm{op}} \subset \mathbf{B}(\mathrm{L}^2(M) \otimes \mathrm{L}^2(M))$, we will denote by $\|\cdot\|_{\min}$ the operator norm on $M \otimes_{\mathrm{alg}} M^{\mathrm{op}}$ induced by $\mathbf{B}(\mathrm{L}^2(M) \otimes \mathrm{L}^2(M))$. It is called the *minimal tensor norm.* We will also denote by $M \mathbin{\overline{\otimes}} M^{\mathrm{op}} := (M \otimes_{\mathrm{alg}} M^{\mathrm{op}})'' \subset \mathbf{B}(\mathrm{L}^2(M) \otimes \mathrm{L}^2(M))$.

Let H be a separable Hilbert space. For every $p \geq 1$, define the pth-*Schatten class* $\mathcal{S}_p(H)$ by

$$\mathcal{S}_p(H) = \{T \in \mathbf{B}(H) : \mathrm{Tr}(|T|^p) < \infty\}.$$

It is a Banach space with norm given by $\|T\|_p = \mathrm{Tr}(|T|^p)^{1/p}$. Observe that $\mathcal{S}_1(H)$ is the space of *trace-class* operators and $\mathcal{S}_2(H)$ is the (Hilbert) space of Hilbert-Schmidt operators. It is also denoted by $\mathrm{HS}(H)$.

Let M be a finite von Neumann algebra with a distinguished faithful normal trace τ. Observe that the unitary $U : \mathrm{HS}(\mathrm{L}^2(M)) \to \mathrm{L}^2(M) \otimes \mathrm{L}^2(M)$ defined by $U(\langle \cdot, \eta\rangle \xi) = \xi \otimes J\eta$ is an M-M-bimodule isomorphism.

We will be using the following technical results.

Lemma 4.6. *Let A be a unital C^*-algebra, $u \in (A)_1$ and $\omega \in A^*$ a state. Then we have*

$$\max\{\|\omega - \omega(u\,\cdot\,)\|, \|\omega - \omega(\,\cdot\, u^*)\|, \|\omega - \omega \circ \mathrm{Ad}(u)\|\} \leq 2\sqrt{2|1 - \omega(u)|}.$$

Proof. Let $(\pi_\omega, \mathcal{H}_\omega, \xi_\omega)$ the GNS representation associated with the state ω on A. Then $\omega(a) = \langle \pi_\omega(a)\xi_\omega, \xi_\omega\rangle$ for all $a \in A$. We have

$$\|\omega - \omega(\,\cdot\, u^*)\| \leq \|\xi_\omega - \pi_\omega(u)^*\xi_\omega\| \leq \sqrt{2(1 - \Re\omega(u))} \leq \sqrt{2|1 - \omega(u)|}.$$

Likewise, we get $\|\omega - \omega(u\,\cdot\,)\| \leq \sqrt{2|1 - \omega(u)|}$. Moreover, we have

$$\|\omega - \omega \circ \mathrm{Ad}(u)\| \leq 2\|\xi_\omega - \pi_\omega(u)^*\xi_\omega\| \leq 2\sqrt{2|1 - \omega(u)|}.$$ □

The previous lemma implies in particular that when $\omega(u) = 1$, then

$$\omega = \omega(\,\cdot\, u^*) = \omega(u\,\cdot\,) = \omega \circ \mathrm{Ad}(u).$$

Lemma 4.7 (Powers–Størmer's Inequality)**.** *Let H be a Hilbert space and $S, T \in \mathcal{S}_2(H)_+$. Then we have*

$$\|S - T\|_2^2 \leq \|S^2 - T^2\|_1 \leq \|S - T\|_2 \|S + T\|_2.$$

Before starting the proof, we make the following observations:

- Whenever $A, B \in \mathbf{B}(H)$ have finite rank and if we write $AB = U|AB|$ for the polar decomposition, by the Cauchy–Schwarz Inequality, we have

$$\|AB\|_1 = \mathrm{Tr}(|AB|) = \mathrm{Tr}(U^*AB) \leq \|U^*A\|_2\|B\|_2 \leq \|A\|_2\|B\|_2.$$

- Whenever $A, B \in \mathbf{B}(H)_+$ and A or B has finite rank, we have $\mathrm{Tr}(AB) \geq 0$. Indeed, without loss of generality, we may assume that B has finite rank and we write $B = \sum_{i=1}^n \lambda_i \langle \cdot, \xi_i\rangle\, \xi_i$. Then $AB = \sum_{i=1}^n \lambda_i \langle \cdot, \xi_i\rangle\, A\xi_i$ and so $\mathrm{Tr}(AB) = \sum_{i=1}^n \lambda_i \langle A\xi_i, \xi_i\rangle \geq 0$.

Proof. We reproduce the elegant proof given in [BO08, Proposition 6.2.4]. First observe that using the Spectral Theorem, we may assume that S, T have both finite rank and still satisfy $S, T \geq 0$.

The identity

$$S^2 - T^2 = \frac{1}{2}((S+T)(S-T) + (S-T)(S+T)) \tag{4.1}$$

together with the first observation give the right inequality.

Put $p = \mathbf{1}_{[0,+\infty)}(S-T)$. We have $(S-T)p \geq 0$ and $(T-S)p^\perp \geq 0$. Observe that we also have

$$\begin{aligned} \operatorname{Tr}((S+T)(S-T)p) &= \operatorname{Tr}((S+T)p(S-T)) \\ &= \operatorname{Tr}((S-T)(S+T)p) \end{aligned} \tag{4.2}$$

$$\begin{aligned} \operatorname{Tr}((T+S)(T-S)p^\perp) &= \operatorname{Tr}((T+S)p^\perp(T-S)) \\ &= \operatorname{Tr}((T-S)(T+S)p^\perp). \end{aligned} \tag{4.3}$$

Then we have

$$\begin{aligned} \|S-T\|_2^2 &= \operatorname{Tr}((S-T)^2) \\ &= \operatorname{Tr}((S-T)^2 p + (S-T)^2 p^\perp) \\ &= \operatorname{Tr}((S-T)(S-T)p + (T-S)(T-S)p^\perp) \\ &\leq \operatorname{Tr}((S+T)(S-T)p + (T+S)(T-S)p^\perp) \quad \text{(using the second obsevation)} \\ &= \operatorname{Tr}((S^2-T^2)p + (T^2-S^2)p^\perp) \quad \text{(using (4.1), (4.2) and (4.3))} \\ &\leq \operatorname{Tr}(|S^2-T^2|p + |T^2-S^2|p^\perp) \quad \text{(using the second observation)} \\ &= \operatorname{Tr}(|S^2-T^2|) = \|S^2-T^2\|_1. \end{aligned}$$

□

4.3. **Connes's fundamental theorem.** This section is devoted to proving Connes's characterization of *amenability* for tracial von Neumann algebras.

Definition 4.8. Let $M \subset \mathbf{B}(H)$ be any von Neumann algebra with separable predual. We say that

- M is *amenable* if there exists a conditional expectation $\Phi : \mathbf{B}(H) \to M$.
- M is *hyperfinite* if there exists an increasing sequence of unital finite dimensional $*$-subalgebras $Q_n \subset M$ such that $M = \bigvee_n Q_n$.

Theorem 4.9 (Connes [Co75])**.** *Let (M, τ) be a tracial von Neumann algebra with separable predual. The following are equivalent:*

(1) There exists a conditional expectation $\Phi : \mathbf{B}(\mathrm{L}^2(M)) \to M$.
(2) There exists an M-central state φ on $\mathbf{B}(\mathrm{L}^2(M))$ such that $\varphi|_M = \tau$.
(3) There exists a net of unit vectors $\xi_n \in \mathrm{L}^2(M) \otimes \mathrm{L}^2(M)$ such that $\lim_n \|x\xi_n - \xi_n x\|_2 = 0$ and $\lim_n \langle x\xi_n, \xi_n\rangle = \tau(x)$ for all $x \in M$.
(4) ${}_M\mathrm{L}^2(M)_M \subset_{\text{weak}} {}_M(\mathrm{L}^2(M) \otimes \mathrm{L}^2(M))_M$.
(5) For all $a_1, \dots, a_k, b_1, \dots, b_k \in M$, we have

$$\left|\tau\left(\sum_{i=1}^k a_i b_i\right)\right| \leq \left\|\sum_{i=1}^k a_i \otimes b_i^{\mathrm{op}}\right\|_{\min}.$$

(6) M is hyperfinite.

Whenever $M = \mathrm{L}(\Gamma)$ is the von Neumann algebra of a countable discrete group, the previous conditions are equivalent to:

(7) Γ *is amenable.*

Proof. We show that (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) $\Leftrightarrow$ (7) and (6) $\Rightarrow$ (1). The proof of (1) $\Rightarrow$ (6) is beyond the scope of these notes.

(1) $\Rightarrow$ (2) Put $\varphi = \tau \circ \Phi$.

(2) $\Rightarrow$ (3) Let φ be an M-central state on $\mathbf{B}(\mathrm{L}^2(M))$. Since the set of normal states is $\sigma(\mathbf{B}(\mathrm{L}^2(M))^*, \mathbf{B}(\mathrm{L}^2(M)))$-dense in the set of states, we may choose a net of normal states $(\varphi_j)_{j\in J}$ on $\mathbf{B}(\mathrm{L}^2(M))$ such that $\lim_J \varphi_j(T) = \varphi(T)$ for all $T \in \mathbf{B}(\mathrm{L}^2(M))$. We get that $\varphi_j \circ \mathrm{Ad}(u) - \varphi_j \to 0$ with respect to the $\sigma(\mathbf{B}(\mathrm{L}^2(M))_*, \mathbf{B}(\mathrm{L}^2(M)))$-topology, for all $u \in \mathcal{U}(M)$. Using Hahn–Banach Theorem and up to replacing the net $(\varphi_j)_{j\in J}$ by a net $(\varphi'_k)_{k\in K}$ where each φ'_k is equal to a finite convex combination of some of the φ_j's, we may assume that $\|\varphi_j \circ \mathrm{Ad}(u) - \varphi_j\| \to 0$ for all $u \in \mathcal{U}(M)$. For every $j \in J$, let $T_j \in \mathcal{S}_1(\mathrm{L}^2(M))_+$ be the unique trace-class operator such that $\varphi_j(S) = \mathrm{Tr}(T_j S)$ for all $S \in \mathbf{B}(\mathrm{L}^2(M))$. We get $\|T_j\|_1 = 1$ and $\lim_J \|uT_j u^* - T_j\|_1 = 0$ for all $u \in \mathcal{U}(M)$. Put $\xi_j = T_j^{1/2} \in \mathcal{S}_2(\mathrm{L}^2(M))$ and observe that $\|\xi_j\|_2 = 1$. Since ξ_j is a Hilbert-Schmidt operator, we may regard $\xi_j \in \mathrm{L}^2(M) \otimes \mathrm{L}^2(M)$. By the Powers-Størmer Inequality, we get $\lim_J \|u\xi_j u^* - \xi_j\|_2 = 0$ for all $u \in \mathcal{U}(M)$. Moreover, we have

$$\lim_J \langle x\xi_j, \xi_j\rangle = \lim_J \mathrm{Tr}(T_j x) = \lim_J \varphi_j(x) = \varphi(x) = \tau(x), \forall x \in M.$$

(3) $\Rightarrow$ (4) Let $a_1, \dots, a_k, b_1, \dots, b_k \in M$ and put $T = \sum_{i=1}^k a_i \otimes b_i{}^{\mathrm{op}}$. Let $c, d \in M$. Then

$$\begin{aligned}
\left|\langle \pi_{\mathrm{L}^2(M)}(T) c\xi_\tau, d\xi_\tau\rangle\right| &= \left|\tau\left(\sum_{i=1}^k d^* a_i c b_i\right)\right| \\
&= \lim_n \left|\left\langle \sum_{i=1}^k d^* a_i c b_i \, \xi_n, \xi_n \right\rangle\right| \\
&= \lim_n \left|\left\langle \sum_{i=1}^k a_i \, \xi_n c \, b_i, d\xi_n \right\rangle\right| \\
&\leq \|\pi_{\mathrm{L}^2(M)\otimes \mathrm{L}^2(M)}(T)\| \, \lim_n \|\xi_n c\| \lim_n \|d\xi_n\| \\
&= \|\pi_{\mathrm{L}^2(M)\otimes \mathrm{L}^2(M)}(T)\| \, \|c\|_2 \|d\|_2.
\end{aligned}$$

This implies that $\|\pi_{\mathrm{L}^2(M)}(T)\| \leq \|\pi_{\mathrm{L}^2(M)\otimes \mathrm{L}^2(M)}(T)\|$.

(4) $\Rightarrow$ (5) Let $a_1, \dots, a_k, b_1, \dots, b_k \in M$ and put $T = \sum_{i=1}^k a_i \otimes b_i^{\mathrm{op}}$. Since $\mathrm{L}^2(M) \otimes \mathrm{L}^2(M)$ is a left $M \mathbin{\overline{\otimes}} M^{\mathrm{op}}$-module, we have

$$\|\pi_{\mathrm{L}^2(M)\otimes \mathrm{L}^2(M)}(T)\| = \left\|\sum_{i=1}^k a_i \otimes b_i^{\mathrm{op}}\right\|_{\min}.$$

Since by assumption, we have $\|\pi_{\mathrm{L}^2(M)}(T)\| \leq \|\pi_{\mathrm{L}^2(M)\otimes \mathrm{L}^2(M)}(T)\|$, we get

$$\left|\tau\left(\sum_{i=1}^k a_i b_i\right)\right| = \left|\left\langle \pi_{\mathrm{L}^2(M)}(T)\xi_\tau, \xi_\tau \right\rangle\right| \leq \|\pi_{\mathrm{L}^2(M)}(T)\| \leq \left\|\sum_{i=1}^k a_i \otimes b_i^{\mathrm{op}}\right\|_{\min}.$$

(5) $\Rightarrow$ (2) Denote by $\Omega : M \otimes_{\rm alg} M^{\rm op} \to \mathbf{C}$ the $\|\cdot\|_{\min}$-bounded functional such that $\Omega(a \otimes b^{\rm op}) = \tau(ab)$. By the Hahn–Banach Theorem and since $M \otimes_{\rm alg} M^{\rm op} \subset \mathbf{B}(\mathrm{L}^2(M) \otimes \mathrm{L}^2(M))$, we may extend the functional Ω to $\mathbf{B}(\mathrm{L}^2(M) \otimes \mathrm{L}^2(M))$ without increasing the norm of Ω. We still denote this extension by Ω. Since $\|\Omega\| = 1 = \Omega(1)$, Ω is a state on $\mathbf{B}(\mathrm{L}^2(M) \otimes \mathrm{L}^2(M))$. Since $\Omega(u \otimes \overline{u}) = \tau(uu^*) = 1$ for all $u \in \mathcal{U}(M)$, we have

$$\Omega(S(u \otimes \overline{u})) = \Omega(S) = \Omega((u \otimes \overline{u})S)$$

for all $S \in \mathbf{B}(\mathrm{L}^2(M) \otimes \mathrm{L}^2(M))$ and all $u \in \mathcal{U}(M)$ (see Lemma 4.6).

Put $\varphi(T) = \Omega(T \otimes 1^{\rm op})$ for all $T \in \mathbf{B}(\mathrm{L}^2(M))$. Observe that $\varphi(x) = \Omega(x \otimes 1^{\rm op}) = \tau(x)$ for all $x \in M$. Moreover, for all $T \in \mathbf{B}(\mathrm{L}^2(M))$ and all $u \in \mathcal{U}(M)$, we have

$$\begin{aligned} \varphi(uT) &= \Omega(uT \otimes 1^{\rm op}) = \Omega((u \otimes \overline{u})(T \otimes u^{\rm op})) \\ &= \Omega((T \otimes u^{\rm op})(u \otimes \overline{u})) = \Omega(Tu \otimes 1^{\rm op}) \\ &= \varphi(Tu). \end{aligned}$$

(2) $\Rightarrow$ (1) For all $T \in \mathbf{B}(\mathrm{L}^2(M))$, define a sesquilinear form $\kappa_T : \mathrm{L}^2(M) \times \mathrm{L}^2(M) \to \mathbf{C}$ by the formula

$$\kappa_T(x\xi_\tau, y\xi_\tau) = \varphi(y^*Tx).$$

By Cauchy–Schwarz inequality, we have $|\kappa_T(x\xi_\tau, y\xi_\tau)| \leq \|T\|_\infty \|x\|_2 \|y\|_2$ for all $x, y \in M$ and hence there exists $\Phi(T) \in \mathbf{B}(\mathrm{L}^2(M))$ such that $\kappa_T(x\xi_\tau, y\xi_\tau) = \langle \Phi(T)x\xi_\tau, y\xi_\tau\rangle$ for all $x, y \in M$. Observe that $\|\Phi(T)\| \leq \|T\|$. For all $x, y, a \in M$, we have

$$\begin{aligned} \langle \Phi(T)Ja^*J\, x\xi_\tau, y\xi_\tau\rangle &= \langle \Phi(T)xa\xi_\tau, y\xi_\tau\rangle \\ &= \varphi(y^*Txa) \\ &= \varphi((ya^*)^*Tx) \\ &= \langle \Phi(T)x\xi_\tau, ya^*\xi_\tau\rangle \\ &= \langle \Phi(T)x\xi_\tau, JaJy\xi_\tau\rangle \\ &= \langle Ja^*J\Phi(T)\, x\xi_\tau, y\xi_\tau\rangle. \end{aligned}$$

This implies that $\Phi(T) \in (JMJ)' = M$. It is routine to check that $\Phi : \mathbf{B}(\mathrm{L}^2(M)) \to M$ is a conditional expectation.

(6) $\Rightarrow$ (1) Assume that $M = \bigvee_n Q_n$ with $Q_n \subset M$ an increasing sequence of unital finite dimensional $*$-subalgebras. Denote by μ_n the unique Haar probability measure on the compact group $\mathcal{U}(Q_n)$. Choose a nonprincipal ultrafilter ω on $\mathbf{N}$. For all $T \in \mathbf{B}(\mathrm{L}^2(M))$, put

$$\mathrm{E}(T) = \lim_{n \to \omega} \int_{\mathcal{U}(Q_n)} uTu^* \, \mathrm{d}\mu_n(u).$$

Then $\Phi : \mathbf{B}(\mathrm{L}^2(M)) \to M$ defined by $\Phi(T) = J\mathrm{E}(T)J$ is a conditional expectation.

Put $M = \mathrm{L}(\Gamma)$ and denote by $\lambda_s \in M$ the canonical unitaries.

(1) $\Rightarrow$ (7) Let $\varphi \in \mathbf{B}(\ell^2(\Gamma))^*$ be an $\mathrm{L}(\Gamma)$-central state such that $\varphi|_{\mathrm{L}(\Gamma)} = \tau$. Define a state $m \in \ell^\infty(\Gamma)^*$ by $m = \varphi|_{\ell^\infty(\Gamma)}$. Then m is an invariant mean and Γ is amenable.

(7) $\Rightarrow$ (1) Assume that there exists a sequence of unit vectors $\zeta_n \in \ell^2(\Gamma)$ such that $\|\lambda_s\zeta_n - \zeta_n\| = 0$ for all $s \in \Gamma$. Put $M = \mathrm{L}(\Gamma)$. Consider the M-M-bimodule $\mathcal{H}_\lambda$ as defined before. Recall that ${}_M\mathcal{H}_{\lambda M} \cong {}_M(\mathrm{L}^2(M) \otimes \mathrm{L}^2(M))_M$. Put $\xi_n = \zeta_n \otimes \xi_\tau$ and regard $\xi_n \in \mathrm{HS}(\mathrm{L}^2(M))$. Observe that $\lim_n \|\lambda_s\xi_n - \xi_n\lambda_s\| = 0$ all $s \in \Gamma$ and $\langle \lambda_s\xi_n, \xi_n\rangle = \tau(\lambda_s)$ for all $n \in \mathbf{N}$ and all $s \in \Gamma$. This further implies that $\langle x\xi_n, \xi_n\rangle = \tau(x)$ for all $n \in \mathbf{N}$ and all $x \in M$.

Choose a nonprincipal ultrafilter ω on $\mathbf{N}$ and put $\varphi(T) = \lim_{n\to\omega}\langle T\xi_n, \xi_n\rangle$ for all $T \in \mathbf{B}(\mathrm{L}^2(M))$. We have $\varphi(\lambda_s T) = \varphi(T\lambda_s)$ for all $T \in \mathbf{B}(\mathrm{L}^2(M))$ and all $s \in \Gamma$ and $\varphi|_M = \tau$. Let $x \in M$ and write $x = \sum_{s\in\Gamma} x_s\lambda_s$ for its Fourier expansion. Put $x_{\mathcal{F}} = \sum_{s\in\mathcal{F}} x_s\lambda_s \in \mathbf{C}[\Gamma]$ for $\mathcal{F} \subset \Gamma$ finite subset. By Cauchy–Schwarz Inequality, we have

$$|\varphi((x - x_{\mathcal{F}})T)| \leq \varphi((x - x_{\mathcal{F}})(x - x_{\mathcal{F}})^*)^{1/2}\, \varphi(T^*T)^{1/2} = \|x - x_{\mathcal{F}}\|_2\, \varphi(T^*T)^{1/2}$$

and so $\lim_{\mathcal{F}} \varphi(x_{\mathcal{F}}T) = \varphi(xT)$. Likewise, we have $\lim_{\mathcal{F}} \varphi(Tx_{\mathcal{F}}) = \varphi(Tx)$. This implies that $\varphi(xT) = \varphi(Tx)$ for all $x \in M$ and all $T \in \mathbf{B}(\mathrm{L}^2(M))$. □

We say that a tracial von Neumann algebra (M, τ) is *diffuse* if there exists a sequence of unitaries $u_n \in \mathcal{U}(M)$ such that $u_n \to 0$ σ-weakly. One can show that M is diffuse if and only if M has no nonzero minimal projection.

We record the following well-known fact.

Proposition 4.10. *Let $M \subset \mathbf{B}(H)$ be any diffuse tracial von Neumann algebra. Then for any M-central state $\varphi \in \mathbf{B}(H)^*$ we have $\varphi|_{\mathbf{K}(H)} = 0$.*

Proof. Fix a sequence of unitaries $u_n \in \mathcal{U}(M)$ such that $u_n \to 0$ σ-weakly. For any $\xi \in H$, denote by $e_\xi : H \to \mathbf{C}\xi$ the corresponding orthogonal projection. Since $\varphi \in \mathbf{B}(H)^*$ is M-central, we have $\varphi(e_{u_k\xi}) = \varphi(u_k e_\xi u_k^*) = \varphi(e_\xi)$ for every $k \in \mathbf{N}$ and every $\xi \in H$. Write $\|T\|_\varphi = \varphi(T^*T)^{1/2}$ for every $T \in \mathbf{B}(H)$.

Fix $\xi \in H$ and $N \geq 1$. By Cauchy–Schwarz inequality, we have

$$\varphi(e_\xi) = \frac{1}{N}\sum_{i=1}^N \varphi(e_{u_{k_i}\xi}) = \frac{1}{N}\,\varphi\left(\sum_{i=1}^N e_{u_{k_i}\xi}\right) \leq \frac{1}{N}\left\|\sum_{i=1}^N e_{u_{k_i}\xi}\right\|_\varphi.$$

We may choose $k_1, \ldots, k_N \in \mathbf{N}$ such that $\|e_{u_{k_j}\xi}\, e_{u_{k_i}\xi}\|_\infty = |\langle u_{k_j}\xi, u_{k_i}\xi\rangle| \leq \frac{1}{N}$ for all $1 \leq i < j \leq N$. Then we also have

$$\begin{aligned}\left\|\sum_{i=1}^N e_{u_{k_i}\xi}\right\|_\varphi^2 &= \sum_{i=1}^N \varphi(e_{u_{k_i}\xi}) + \sum_{1\leq i\neq j\leq N} \varphi(e_{u_{k_j}\xi}\, e_{u_{k_i}\xi}) \\ &\leq N + 2\sum_{1\leq i<j\leq N} \|e_{u_{k_j}\xi}\, e_{u_{k_i}\xi}\|_\infty \\ &\leq N + N(N-1)\frac{1}{N} = 2N - 1.\end{aligned}$$

Thus, we obtain

$$\varphi(e_\xi) \leq \frac{\sqrt{2N-1}}{N}.$$

Since this holds for every $N \geq 1$, it follows that $\varphi(e_\xi) = 0$. By Cauchy–Schwarz inequality, we also have $\varphi(Se_\xi) = 0$ for every $S \in \mathbf{B}(H)$. It follows that $\varphi(T) = 0$ for every rank one operator $T \in \mathbf{B}(H)$ and hence $\varphi|_{\mathbf{K}(H)} = 0$. □

Exercise 4.11. Let $\Gamma \curvearrowright (X, \mu)$ be a pmp action of a countable discrete group on a standard probability space. Show that $\mathrm{L}^\infty(X) \rtimes \Gamma$ is amenable if and only if Γ is amenable.

Exercise 4.12. Let $A \subset M$ be any inclusion of tracial von Neumann algebras. Assume that A is amenable. Show that for every $u \in \mathcal{N}_M(A)$, the von Neumann subalgebra $\langle A, u\rangle \subset M$ is amenable.

5. Strong solidity of free group factors

5.1. **Free groups are bi-exact.** Recall that a countable discrete group Γ is amenable if and only if any action $\Gamma \curvearrowright X$ on any metrizable compact space admits an invariant probability measure. We introduce a generalization of this notion to actions on compact spaces as follows. Put $\mathrm{Prob}(\Gamma) := \left\{\mu \in \ell^1(\Gamma) : \mu \geq 0 \text{ and } \|\mu\|_1 = 1\right\} \subset \ell^1(\Gamma)$.

Definition 5.1. Let Γ be any countable discrete group, X any metrizable compact space and $\Gamma \curvearrowright X$ any action by homeomorphisms. We say that the action $\Gamma \curvearrowright X$ is *topologically amenable* if there exists a sequence of continuous maps $\mu_k : X \to \mathrm{Prob}(\Gamma)$ such that

$$\lim_{k\to\infty} \left(\sup_{x \in X} \|s\mu_k(x) - \mu_k(sx)\|_1 \right) = 0$$

for all $s \in \Gamma$.

Note that conitinuity of μ_k in Definition 5.1 means that for any convergent net $x_i \to x$ in X we have $\mu_k(x_i)(s) \to \mu_k(x)(s)$ for every $s \in \Gamma$.

The next definition will be central in this section.

Definition 5.2 (Brown–Ozawa [BO08])**.** A countable discrete group Γ is said to be *bi-exact* if Γ admits a compactification $\Gamma \subset X$ such that the left-right action $\Gamma \times \Gamma \curvearrowright \Gamma$ extends to an action by homeomorphisms $\Gamma \times \Gamma \curvearrowright X$ which satisfies the following properties:

(1) The left action $\Gamma \curvearrowright X$ is topologically amenable.
(2) The right action $\Gamma \curvearrowright X \setminus \Gamma$ is trivial.

Proposition 5.3. *Free groups are bi-exact.*

Proof. Let $n \geq 2$ and regard $\mathbf{F}_n = \langle g_1, \dots, g_n \rangle$. Define the *boundary* of $\mathbf{F}_n$ by

$$\partial \mathbf{F}_n := \left\{ (a_k)_k \in \prod_{\mathbf{N}} \left\{g_1, g_1^{-1}, \dots, g_n, g_n^{-1}\right\} : a_k^{-1} \neq a_{k+1}, \forall k \in \mathbf{N} \right\}.$$

Denote by $\ell : \mathbf{F}_n \to \mathbf{N}$ the canonical length. Endowed with the relative product topology, $\partial \mathbf{F}_n$ is a compact space and with an appropriate topology, $X := \mathbf{F}_n \cup \partial \mathbf{F}_n$ is a compactification of $\mathbf{F}_n$. For every $x = (a_k)_k \in X$ and every $j \in \mathbf{N}$, define $x(j) = a_0 \cdots a_j \in \mathbf{F}_n$. For every $k \in \mathbf{N}$, the map

$$\mu_k : X \to \mathrm{Prob}(\mathbf{F}_n) : x \mapsto \frac{1}{k+1} \sum_{j=0}^{k} \delta_{x(j)}$$

is continuous and satisfies

$$\sup_{x \in X} \|s\mu_k(x) - \mu_k(sx)\| \leq \frac{2\ell(s)}{k+1}$$

for all $s \in \mathbf{F}_n$ and all $k \in \mathbf{N}$. This shows that the left action $\mathbf{F}_n \curvearrowright X$ is amenable. It is clear that the right action $\mathbf{F}_n \curvearrowright \partial \mathbf{F}_n$ is trivial. ☐

5.2. **Free groups have the complete metric approximation property.**

Definition 5.4. Let A, B be any unital C*-algebras and $\varphi : A \to B$ any linear map. We say that φ is *completely bounded* if

$$\|\varphi\|_{\mathrm{cb}} := \sup_{n \geq 1} \|\varphi_n : \mathbf{M}_n(A) \to \mathbf{M}_n(B) : [a_{i,j}]_{i,j} \mapsto [\varphi(a_{i,j})]_{i,j}\| < \infty.$$

We say that φ is *completely contractive* if $\|\varphi\|_{\mathrm{cb}} \leq 1$.

Proposition 5.5. *Let A be any unital C*-algebra, $\pi : A \to \mathbf{B}(K)$ any unital $*$-representation and $V, W : H \to K$ any isometries. Then the linear map $\varphi : A \to \mathbf{B}(H) : a \mapsto V^*\pi(a)W$ is completely contractive.*

Proof. Let $n \geq 1$. Observe that $\mathbf{M}_n(\mathbf{B}(H)) = \mathbf{B}(H^{\oplus n})$ and $\mathbf{M}_n(\mathbf{B}(K)) = \mathbf{B}(K^{\oplus n})$. Denote by $V^{(n)}, W^{(n)} = H^{\oplus n} \to K^{\oplus n}$ the canonical amplifications. Observe moreover that $\pi_n = \mathrm{id}_{\mathbf{M}_n(\mathbf{C})} \otimes \pi : \mathbf{M}_n(A) \to \mathbf{M}_n(\mathbf{B}(K))$ is a unital $*$-homomorphism. For every $a \in \mathbf{M}_n(A)$, we have

$$\varphi_n(a) = (V^{(n)})^* \, \pi_n(a) \, W^{(n)}$$

and hence $\varphi_n : \mathbf{M}_n(A) \to \mathbf{M}_n(\mathbf{B}(H))$ is a contraction. This shows that φ is completely contractive. $\square$

One can prove that any completely contractive map $\varphi : A \to \mathbf{B}(H)$ admits a decomposition as in Proposition 5.5.

Let Γ be any countable discrete group. A function $\varphi : \Gamma \to \mathbf{C}$ is said to be a *Herz-Schur multiplier* if the linear map

$$\mathfrak{m}_\varphi : \mathbf{B}(\ell^2(\Gamma)) \to \mathbf{B}(\ell^2(\Gamma)) : [T_{s,t}]_{s,t} \mapsto [\varphi(s^{-1}t) \, T_{s,t}]_{s,t}$$

is well defined, ultraweakly continuous and completely bounded. Observe that in this case, we have $\mathfrak{m}_\varphi(\lambda_s) = \varphi(s)\lambda_s$ for every $s \in \Gamma$. Therefore, the restriction map $\mathfrak{m}_\varphi : \mathrm{L}(\Gamma) \to \mathrm{L}(\Gamma) : \lambda_s \mapsto \varphi(s)\lambda_s$ is well defined, ultraweakly continuous and completely bounded. Denote by $\mathrm{B}_2(\Gamma)$ the Banach space of all Herz-Schur multipliers $\varphi : \Gamma \to \mathbf{C}$ endowed with the norm $\|\varphi\|_{\mathrm{B}_2} := \|\mathfrak{m}_\varphi\|_{\mathrm{cb}}$.

Proposition 5.6. *Let Γ be any countable discrete group and $\varphi : \Gamma \to \mathbf{C}$ any function for which there exist a Hilbert space H and families $(\xi_s)_s$ and $(\eta_t)_t$ in H with $\sup_{s \in \Gamma} \|\xi_s\| \leq 1$ and $\sup_{t \in \Gamma} \|\eta_t\| \leq 1$ such that $\varphi(s^{-1}t) = \langle \eta_t, \xi_s \rangle$ for all $s, t \in \Gamma$. Then φ is a Herz-Schur multiplier with $\|\varphi\|_{\mathrm{B}_2} \leq 1$.*

Proof. Define contractions $V, W : \ell^2(\Gamma) \to \ell^2(\Gamma) \otimes H$ by $V(\delta_s) = \delta_s \otimes \xi_s$ and $W(\delta_t) = \delta_t \otimes \eta_t$ for all $s, t \in \Gamma$. A simple calculation shows that $\mathfrak{m}_\varphi(e_{s,t}) = V^*(e_{s,t} \otimes 1)W$ for all $s, t \in \Gamma$. Therefore, we have $\mathfrak{m}_\varphi(T) = V^*(T \otimes 1)W$ for all $T \in \mathbf{B}(\ell^2(\Gamma))$. By Proposition 5.5, φ is a Herz-Schur multiplier. $\square$

Corollary 5.7. *For any $\varphi \in \ell^2(\Gamma)$, we have $\varphi \in \mathrm{B}_2(\Gamma)$ and $\|\varphi\|_{\mathrm{B}_2} \leq \|\varphi\|_2$. Thus, the Banach subspace $F \subset \mathrm{B}_2(\Gamma)$ generated by finitely supported functions contains $\ell^2(\Gamma)$.*

Proof. Let $\varphi \in \ell^2(\Gamma)$. For any $s, t \in \Gamma$, put $\xi_s := \delta_{s^{-1}}$ and $\eta_t := \rho_t(\varphi) \in \ell^2(\Gamma)$. Observe that $\sup_{s \in \Gamma} \|\xi_s\|_2 = 1$ and $\sup_{t \in \Gamma} \|\eta_t\|_2 = \|\varphi\|_2$. Moreover, for any $s, t \in \Gamma$, we have $\langle \eta_t, \xi_s \rangle = \langle \rho_t(\varphi), \delta_{s^{-1}} \rangle = \varphi(s^{-1}t)$. Therefore, $\varphi \in \mathrm{B}_2(\Gamma)$ and $\|\varphi\|_{\mathrm{B}_2} \leq \|\varphi\|_2$ by Proposition 5.6. $\square$

Definition 5.8 (Haagerup)**.** Let Γ be any countable discrete group. We say that Γ has the *complete metric approximation property* (CMAP) if there exists a sequence of finitely supported Herz-Schur multipliers $\varphi_n : \Gamma \to \mathbf{C}$ such that $\lim_n \varphi_n = 1$ pointwise and $\lim_n \|\varphi_n\|_{\mathrm{B}_2} = 1$.

Theorem 5.9 (Haagerup)**.** *Free groups have the complete metric approximation property.*

Proof. It suffices to prove the result for $\mathbf{F}_2 = \langle a, b \rangle$. We reproduce the elegant proof given in [BO08, Chapter 12]. We identify $\mathbf{F}_2$ with its canonical Cayley graph which is a 4-regular tree and denote by $\ell : \mathbf{F}_2 \to \mathbf{N}$ the canonical length. Denote by $\omega = \omega_e$ the unique infinite geodesic ray in $\mathbf{F}_2$ which starts at the neutral element and which contains a^k for all $k \in \mathbf{N}$. For any $s \in \mathbf{F}_2$, denote by ω_s the unique infinite geodesic ray in $\mathbf{F}_2$ which starts at s and eventually flows into ω.

Put $\mathbf{D} := \{z \in \mathbf{C} : |z| < 1\}$. For every $z \in \mathbf{D}$ and every $s \in \mathbf{F}_2$, define $\zeta_s(z) \in \ell^2(\mathbf{F}_2)$ by the formula

$$\zeta_s(z) = \sqrt{1 - z^2} \sum_{k=0}^{\infty} z^k \delta_{\omega_s(k)}.$$

The above series converges absolutely in $z \in \mathbf{D}$ and uniformly in $s \in \mathbf{F}_2$ and we have

$$\|\zeta_s(z)\|_2^2 = |1 - z^2| \sum_{k=0}^{\infty} |z|^k = \frac{|1 - z^2|}{1 - |z|^2}.$$

In particular, we have that the fonction $\zeta : \mathbf{D} \to \ell^\infty(\mathbf{F}_2, \ell^2(\mathbf{F}_2)) : z \mapsto (\zeta_s(z))_s$ is holomophic. For every $z \in \mathbf{D}$, put $\varphi_z : \mathbf{F}_2 \to \mathbf{C} : s \mapsto z^{\ell(s)}$. A simple calculation shows that

$$\begin{aligned} \langle \zeta_t(z), \overline{\zeta_s(z)} \rangle &= (1 - z^2) \sum_{k,l=0}^{\infty} z^{k+l} \delta_{\omega_s(k), \omega_t(l)} \\ &= (1 - z^2) \sum_{n=0}^{\infty} z^{\ell(s^{-1}t) + 2n} \\ &= z^{\ell(s^{-1}t)} \\ &= \varphi_z(s^{-1}t). \end{aligned}$$

In particular, the map $\varphi : \mathbf{D} \to \mathrm{B}_2(\mathbf{F}_2) : z \mapsto \varphi_z$ is holomorphic.

Observe that for every $0 \leq r < 1$ and every $s \in \mathbf{F}_2$, $\zeta_s(r) = \overline{\zeta_s(r)}$ and hence φ_r is positive definite. Thus, we have $\|\varphi_r\|_{\mathrm{B}_2} = \varphi_r(e) = 1$. Moreover, we have $\lim_{r \to 1} \varphi_r = 1$ pointwise. It then suffices to show that $\varphi_z \in F$ for every $z \in \mathbf{D}$, where $F \subset \mathrm{B}_2(\mathbf{F}_2)$ is the Banach subspace generated by finitely supported functions. Observe that $\sharp \{s \in \mathbf{F}_2 : \ell(s) \leq n\} = 4^n + 1$. This implies that $\varphi_z \in \ell^1(\mathbf{F}_2) \subset F \subset \mathrm{B}_2(\mathbf{F}_2)$ for any $z \in \mathbf{D}$ such that $|z| < 1/4$ (see Corollary 5.7). Therefore, the map $\varphi_F : \mathbf{D} \to \mathrm{B}_2(\mathbf{F}_2)/F : z \mapsto \varphi_z + F$ is holomorphic and zero for any $z \in \mathbf{D}$ such that $|z| < 1/4$. This shows that $\varphi_F = 0$ and finally implies that $\varphi_z \in F$ for every $z \in \mathbf{D}$. □

5.3. **Ozawa–Popa's weak compactness criterion.** Let Γ be any countable discrete group and (X, μ) any standard probability space. We say that a pmp action $\Gamma \curvearrowright (X, \mu)$ is *compact* when the range of the homomorphism $\sigma : \Gamma \to \mathrm{Aut}(X, \mu)$ is precompact in the Polish group $\mathrm{Aut}(X, \mu)$. For instance, whenever $\Gamma < K$ is a dense subgroup of a compact second countable group, the pmp action by left translation $\Gamma \curvearrowright K$ is compact.

Ozawa–Popa discovered in [OP07] that inside group von Neumann algebras $\mathrm{L}(\Gamma)$ where Γ has the CMAP, the action $\mathcal{N}_M(A) \curvearrowright A$ of the normalizer $\mathcal{N}_M(A)$ of any amenable subalgebra $A \subset \mathrm{L}(\Gamma)$ satisfies a weak form of *compactness*. More precisely, they obtained the following result.

Theorem 5.10 (Ozawa–Popa [OP07])**.** *Let Γ be any countable discrete group with the CMAP and put $M := \mathrm{L}(\Gamma)$. Let $A \subset M$ be any amenable von Neumann subalgebra. Then the trace preserving action $\mathcal{N}_M(A) \curvearrowright A$ is* weakly compact *in the following sense. There exists a state $\varphi \in \mathbf{B}(\mathrm{L}^2(M))^*$ such that*

- $\varphi(aT) = \varphi(Ta)$ *for all $a \in A$ and all $T \in \mathbf{B}(\mathrm{L}^2(M))$.*
- $\varphi(uJuJ\,T) = \varphi(T\,uJuJ)$ *for all $u \in \mathcal{N}_M(A)$ and all $T \in \mathbf{B}(\mathrm{L}^2(M))$.*
- $\varphi(x) = \tau(x) = \varphi(Jx^*J)$ *for all $x \in M$.*

Proof. Denote by $\varphi_n : M \to M$ a sequence of finite rank normal completely bounded maps that witness CMAP. Define the normal linear functionals $\mu_n : M \mathbin{\overline{\otimes}} M^{\mathrm{op}} \to \mathbf{C}$ by the formula

$$\mu_n(a \otimes b^{\mathrm{op}}) = \tau(\varphi_n(a)b), \forall a, b \in M.$$

Let $A \subset Q \subset M$ be any intermediate amenable von Neumann subalgebra. Put $\mu_n^Q = \mu_n|_{Q \overline{\otimes} Q^{\mathrm{op}}}$. Using Theorem 4.9(5), we know that for all $k \geq 1$ and all $a_1, \dots, a_k, b_1, \cdots, b_k \in Q$, we have

$$\begin{aligned}
\left|\tau\left(\sum_{i=1}^k \varphi_n(a_i)b_i\right)\right| &= \left|\tau\left(\sum_{i=1}^k \mathrm{E}_Q(\varphi_n(a_i))b_i\right)\right| \\
&\leq \left\|\sum_{i=1}^k \mathrm{E}_Q(\varphi_n(a_i)) \otimes b_i^{\mathrm{op}}\right\|_{\min} \\
&\leq \left\|\sum_{i=1}^k \varphi_n(a_i) \otimes b_i^{\mathrm{op}}\right\|_{\min} \\
&= \left\|(\varphi_n \otimes \mathrm{id}_{Q^{\mathrm{op}}}) \sum_{i=1}^k a_i \otimes b_i^{\mathrm{op}}\right\|_{\min} \\
&\leq \|\varphi_n\|_{\mathrm{cb}} \cdot \left\|\sum_{i=1}^k a_i \otimes b_i^{\mathrm{op}}\right\|_{\min}.
\end{aligned}$$

This implies that $\|\mu_n^Q\| \leq \|\varphi_n\|_{\mathrm{cb}}$. Write $\mu_n^Q = u_n|\mu_n^Q|$ for the polar decomposition of $\mu_n^Q \in (Q \mathbin{\overline{\otimes}} Q^{\mathrm{op}})_*$. We have $u_n \in (Q \mathbin{\overline{\otimes}} Q^{\mathrm{op}})_1$. Put $\omega_n^Q = \|\mu_n^Q\|^{-1}|\mu_n^Q|$ so that that ω_n^Q is a normal state on $Q \mathbin{\overline{\otimes}} Q^{\mathrm{op}}$. We have $\mu_n^Q = \|\mu_n^Q\|u_n\,\omega_n^Q$. Since $\lim_n \|\mu_n^Q\| = 1$ and $\lim_n \mu_n(1 \otimes 1^{\mathrm{op}}) = 1$, Lemma 4.6 implies that

$$\lim_n \|\omega_n^Q - \mu_n^Q\| = 0. \tag{5.1}$$

Now consider the case when $Q = A$. For all $a \in \mathcal{U}(A)$, since $\mu_n^A(a \otimes \bar{a}) = \tau(\varphi_n(a)a^*) \to 1$, Equation (5.1) and Lemma 4.6 imply that

$$\lim_n \|(a \otimes \bar{a})\omega_n^A - \omega_n^A\| = 0 \quad \text{and} \quad \lim_n \|\omega_n^A(a \otimes \bar{a}) - \omega_n^A\| = 0. \tag{5.2}$$

Let $u \in \mathcal{N}_M(A)$. Next consider the case when $Q = \langle A, u\rangle$ which is amenable by Exercise 4.12. Since $\mu_n^Q(u \otimes \bar{u}) = \tau(\varphi_n(u)u^*) \to 1$, Equation (5.1) and Lemma 4.6 imply that

$$\lim_n \|\mu_n^Q - \mu_n^Q \circ \mathrm{Ad}(u \otimes \bar{u})\| = 0.$$

Since $\mu_n^Q|_{A \overline{\otimes} A^{\mathrm{op}}} = \mu_n^A$ and $\mu_n^Q \circ \mathrm{Ad}(u \otimes \bar{u})|_{A \overline{\otimes} A^{\mathrm{op}}} = \mu_n^A \circ \mathrm{Ad}(u \otimes \bar{u})$, the above equation and Lemma 4.6 imply that

$$\lim_n \|\omega_n^A - \omega_n^A \circ \mathrm{Ad}(u \otimes \bar{u})\| = 0. \tag{5.3}$$

Regard $\omega_n^A \in \mathrm{L}^1(A \mathbin{\overline{\otimes}} A^{\mathrm{op}})_+ \subset \mathrm{L}^1(M \mathbin{\overline{\otimes}} M^{\mathrm{op}})_+$ and put $\xi_n := (\omega_n^A)^{1/2} \in \mathrm{L}^2(M \mathbin{\overline{\otimes}} M^{\mathrm{op}})_+$. The following assertions hold true:

- $\lim_n \|\xi_n - (a \otimes \bar{a})\xi_n\|_2 = 0$ for all $a \in \mathcal{U}(A)$, by (5.2).
- $\lim_n \|\xi_n - (u \otimes \bar{u})\xi_n(u \otimes \bar{u})^*\|_2 = 0$ for all $u \in \mathcal{N}_M(A)$, by (5.3) and Powers-Størmer inequality as in Lemma 4.7.
- $\lim_n \langle (x \otimes 1)\xi_n, \xi_n\rangle = \tau(x)$ and $\lim_n \langle (1 \otimes x^{\mathrm{op}})\xi_n, \xi_n\rangle = \tau(x)$ for all $x \in M$, by construction.

Then choose a nonprincipal ultrafilter $\omega \in \beta(\mathbf{N}) \setminus \mathbf{N}$ and define $\varphi \in \mathbf{B}(\mathrm{L}^2(M))^*$ by the formula

$$\varphi(T) := \lim_{n \to \omega} \langle (T \otimes 1^{\mathrm{op}})\xi_n, \xi_n\rangle, \forall T \in \mathbf{B}(\mathrm{L}^2(M)).$$

Then φ satisfies the conclusion of Theorem 5.10. Indeed for every $a \in A$ and every $T \in \mathbf{B}(\mathrm{L}^2(M))$, using the facts that $\lim_{n\to\omega} \|(a^* \otimes 1^{\mathrm{op}})\xi_n - (1 \otimes \bar{a})\xi_n\|_2 = 0$ and $\lim_{n\to\omega} \|(a \otimes 1^{\mathrm{op}})\xi_n - (1 \otimes a^{\mathrm{op}})\xi_n\|_2 = 0$, we have

$$\begin{aligned}
\varphi(aT) &= \lim_{n \to \omega} \langle (aT \otimes 1^{\mathrm{op}})\xi_n, \xi_n\rangle \\
&= \lim_{n \to \omega} \langle (T \otimes 1^{\mathrm{op}})\xi_n, (a^* \otimes 1^{\mathrm{op}})\xi_n\rangle \\
&= \lim_{n \to \omega} \langle (T \otimes 1^{\mathrm{op}})\xi_n, (1 \otimes \bar{a})\xi_n\rangle \\
&= \lim_{n \to \omega} \langle (T \otimes a^{\mathrm{op}})\xi_n, \xi_n\rangle \\
&= \lim_{n \to \omega} \langle (Ta \otimes 1^{\mathrm{op}})\xi_n, \xi_n\rangle \\
&= \varphi(Ta).
\end{aligned}$$

Next, for every $u \in \mathcal{N}_M(A)$ and every $T \in \mathbf{B}(\mathrm{L}^2(M))$ using the facts that $\lim_{n\to\omega} \|(u^* \otimes 1^{\mathrm{op}})\xi_n(u \otimes 1^{\mathrm{op}}) - (1 \otimes \bar{u})\xi_n(1 \otimes u^{\mathrm{op}})\|_2 = 0$ and $\lim_{n\to\omega} \|(u \otimes 1^{\mathrm{op}})\xi_n(u^* \otimes 1^{\mathrm{op}}) - (1 \otimes u^{\mathrm{op}})\xi_n(1 \otimes \bar{u})\|_2 = 0$, we have

$$\begin{aligned}
\varphi(uJuJ\,T) &= \lim_{n \to \omega} \langle (uJuJ\,T \otimes 1^{\mathrm{op}})\xi_n, \xi_n\rangle \\
&= \lim_{n \to \omega} \langle (T \otimes 1^{\mathrm{op}})\xi_n, (u^*Ju^*J \otimes 1^{\mathrm{op}})\xi_n\rangle \\
&= \lim_{n \to \omega} \langle (T \otimes 1^{\mathrm{op}})\xi_n, (u^* \otimes 1^{\mathrm{op}})\xi_n(u \otimes 1^{\mathrm{op}})\rangle \\
&= \lim_{n \to \omega} \langle (T \otimes 1^{\mathrm{op}})\xi_n, (1 \otimes \bar{u})\xi_n(1 \otimes u^{\mathrm{op}})\rangle \\
&= \lim_{n \to \omega} \langle (T \otimes 1^{\mathrm{op}})(1 \otimes u^{\mathrm{op}})\xi_n(1 \otimes \bar{u}), \xi_n\rangle \\
&= \lim_{n \to \omega} \langle (T \otimes 1^{\mathrm{op}})(u \otimes 1^{\mathrm{op}})\xi_n(u^* \otimes 1^{\mathrm{op}}), \xi_n\rangle \\
&= \lim_{n \to \omega} \langle (T\,uJuJ \otimes 1^{\mathrm{op}})\xi_n, \xi_n\rangle \\
&= \varphi(T\,uJuJ).
\end{aligned}$$

Finally for every $x \in M$, using the fact that $(J \otimes J^{\mathrm{op}})\xi_n = \xi_n$, we have

$$\begin{aligned}
\varphi(x) &= \lim_{n\to\omega} \langle (x \otimes 1^{\mathrm{op}})\xi_n, \xi_n\rangle \\
&= \tau(x) \\
\varphi(Jx^*J) &= \lim_{n\to\omega} \langle (Jx^*J \otimes 1^{\mathrm{op}})\xi_n, \xi_n\rangle \\
&= \lim_{n\to\omega} \langle (x \otimes 1^{\mathrm{op}})(J \otimes J^{\mathrm{op}})\xi_n, (J \otimes J^{\mathrm{op}})\xi_n\rangle \\
&= \lim_{n\to\omega} \langle (x \otimes 1^{\mathrm{op}})\xi_n, \xi_n\rangle \\
&= \varphi(x) \\
&= \tau(x).
\end{aligned}$$

This finishes the proof of Theorem 5.10. □

5.4. **Free group factors are strongly solid.** The main result of this section is the following theorem due to Ozawa–Popa [OP07] in the case of free groups and to Chifan–Sinclair [CS11] in the case of arbitrary bi-exact discrete groups with the CMAP.

Theorem 5.11 ([OP07, CS11])**.** *Let Γ be any bi-exact group with the CMAP. Then $M := \mathrm{L}(\Gamma)$ is* strongly solid *in the following sense. For any amenable diffuse von Neumann subalgebra $A \subset M$, we have that $\mathcal{N}_M(A)''$ remains amenable.*

The proof of Theorem 5.11 consists in two steps. In the first step, we use Ozawa–Popa's weak compactness criterion from Theorem 5.10 to obtain the existence of a state $\varphi \in \mathbf{B}(\mathrm{L}^2(M))^*$ with good invariance properties. In the second step, we use Popa's deformation/rigidity theory to show that $\mathcal{N}_M(A)''$ is amenable.

We present an elegant proof of the second step due to Boutonnet–Carderi [BC14]. Let Γ be any bi-exact group with the CMAP and put $M := \mathrm{L}(\Gamma)$. Consider the compactification $\Gamma \curvearrowright X$ that witnesses bi-exactness. Observe that we have $\mathrm{c}_0(\Gamma) \subset \mathrm{C}(X) \subset \ell^\infty(\Gamma)$. Denote by

$$\mathcal{B} = \mathrm{C}^*(\mathrm{C}(X) \cup \lambda(\Gamma)) \subset \mathbf{B}(\mathrm{L}^2(M)).$$

Since Γ is bi-exact, we moreover have

$$[\mathcal{B}, \mathrm{C}^*_\rho(\Gamma)] \subset \mathrm{C}^*\left(\lambda(\Gamma)\cdot[\mathrm{C}(X), \rho(\Gamma)]\right) \subset \mathrm{C}^*\left(\lambda(\Gamma)\cdot \mathrm{c}_0(\Gamma)\cdot\rho(\Gamma)\right) = \mathbf{K}(\ell^2(\Gamma)) \tag{5.4}$$

where $[\mathcal{X}, \mathcal{Y}] := \{xy - yx : x \in \mathcal{X}, y \in \mathcal{Y}\}$.

Since the action $\Gamma \curvearrowright X$ is topologically amenable, the unital C^*-algebra $\mathcal{B}$ is *nuclear* and for every state $\varphi \in \mathcal{B}^*$, the von Neumann algebra $\pi_\varphi(\mathcal{B})''$ associated with the GNS representation $(\pi_\varphi, H_\varphi, \xi_\varphi)$ is amenable. We refer to [BO08] for proofs of these facts. The next proposition will be crucial.

Proposition 5.12 ([BC14])**.** *Let $\varphi \in \mathbf{B}(\mathrm{L}^2(M))^*$ be any state such that $\varphi|_{\mathrm{C}^*_\lambda(\Gamma)} = \tau$. Define*

$$A_\varphi := \{x \in M : \varphi(xT) = \varphi(Tx), \forall T \in \mathcal{B}\}.$$

Then A_φ is an amenable von Neumann subalgebra.

Proof. Denote by $(\pi_\varphi, H_\varphi, \xi_\varphi)$ the GNS representation of $\mathcal{B}$ with respect to φ. Put $\mathcal{M} := \pi_\varphi(\mathcal{B})''$ and define the *normal* state $\Phi \in \mathcal{M}_*$ by the formula $\Phi(S) = \langle S\xi_\varphi, \xi_\varphi\rangle$. Observe that $\Phi(\pi_\varphi(T)) = \varphi(T)$ for every $T \in \mathcal{B}$. Put $P := \pi_\varphi(\mathrm{C}^*_\lambda(\Gamma))'' \subset \mathcal{M}$. Observe that $\Phi|_P$ is a normal trace and denote by p its support in P (we have $p \in \mathcal{Z}(P)$). Then the map $\iota : \mathrm{C}^*_\lambda(\Gamma) \to Pp$ is a trace preserving $*$-homomorphism which extends to a surjective $*$-isomorphism $\iota : M \to Pp$. We have

$$\begin{aligned}\iota(A_\varphi) &= \{\iota(x) \in Pp : \Phi(\iota(x)\pi_\varphi(T)) = \Phi(\pi_\varphi(T)\iota(x)), \forall T \in \mathcal{B}\} \\ &= Pp \cap \{S \in p\mathcal{M}p : \Phi(ST) = \Phi(TS), \forall T \in p\mathcal{M}p\}\,.\end{aligned}$$

This shows that $\iota(A_\varphi)$ is a von Neumann subalgebra and so is A_φ. Note the support of Φ in $\mathcal{M}$ is less than or equal to p. Since the unital C*-algebra $\mathcal{B}$ is nuclear, the von Neumann algebra $\mathcal{M} = \pi_\varphi(\mathcal{B})''$ is amenable and so is $p\mathcal{M}p$. By the proof of (2) $\Rightarrow$ (1) in Theorem 4.9, there exists a conditional expectation $\mathrm{E} : p\mathcal{M}p \to \iota(A_\varphi)$. Therefore, $\iota(A_\varphi)$ is amenable and so is A_φ. □

Proof of Theorem 5.11. Let $A \subset M$ be any amenable diffuse von Neumann algebra. Choose a state $\varphi \in \mathbf{B}(\mathrm{L}^2(M))^*$ as in Theorem 5.10 and consider the amenable subalgebra A_φ as in Proposition 5.12. We show that $\mathcal{N}_M(A) \subset A_\varphi$ and hence $\mathcal{N}_M(A)'' \subset A_\varphi$ is amenable.

Let $u \in \mathcal{N}_M(A)$ and $T \in \mathcal{B}$. Observe that $\varphi|_{\mathbf{K}(\mathrm{L}^2(M))} = 0$ by Proposition 4.10. Choose a sequence $(x_n)_n$ in $\mathrm{C}^*_\rho(\Gamma)$ such that $x_n \to Ju^*J$ strongly. Since $\varphi|_{\mathrm{R}(\Gamma)}$ is normal, Cauchy–Schwarz inequality implies that

$$\begin{aligned}\lim_n \varphi(uJuJ\,Tx_n) &= \varphi(uJuJ\,TJu^*J) \\ \lim_n \varphi(uJuJ\,x_nT) &= \lim_n \varphi(JuJx_n\,uT) \\ &= \varphi(JuJJu^*J\,uT) \\ &= \varphi(uT).\end{aligned}$$

Since Γ is bi-exact, we know that $uJuJ(x_nT - Tx_n) \in \mathbf{K}(\mathrm{L}^2(M))$ for every $n \in \mathbf{N}$ by (5.4). This implies that $\varphi(uJuJTJu^*J) = \varphi(uT)$. By Theorem 5.10, we have that $\varphi(uJuJ\,TJu^*J) = \varphi(TJu^*J\,uJuJ) = \varphi(Tu)$. This implies that $\varphi(uT) = \varphi(Tu)$ and hence $u \in A_\varphi$. □

Combining Proposition 5.3 and Theorem 5.11, we obtain the following corollary.

Corollary 5.13 (Ozawa–Popa [OP07])**.** *Free group factors are strongly solid.*

The above corollary strenghtens both Voiculescu's result [Vo95] showing that free group factors have no Cartan subalgebra and Ozawa's result [Oz03] showing that free group factors are *solid*, meaning that the relative commutant of any diffuse von Neumann subalgebra is amenable. Recently, Popa–Vaes [PV11] showed that for every free ergodic pmp action $\mathbf{F}_n \curvearrowright (X, \mu)$, $\mathrm{L}^\infty(X) \subset \mathrm{L}^\infty(X) \rtimes \mathbf{F}_n$ is the unique Cartan subalgebra up to unitary conjugacy.

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Laboratoire de Mathématiques d'Orsay, Université Paris-Sud, CNRS, Université Paris-Saclay, 91405 Orsay, FRANCE

E-mail address: `cyril.houdayer@math.u-psud.fr`