AN INVITATION TO VON NEUMANN ALGEBRAS

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Abstract. These are the lectures notes from a minicourse given at the summer school "Rigidity and group actions" at the Institute of Mathematics of Jussieu in June 2013.

Lecture 1

In the first lecture, we review some basic concepts of the theory of von Neumann algebras. These include for instance von Neumann's bicommutant theorem and the GNS construction with respect to a tracial state. As an example, we study group von Neumann algebras.

Basic results on von Neumann algebras. Let H be a (separable) complex Hilbert space. We shall denote by $\langle \cdot, \cdot \rangle$ the inner product on H that we assume to be linear in the first variable and conjugate linear in the second one. Denote by $\mathbf{B}(H)$ the algebra of all bounded linear maps $T : H \to H$. This is a Banach algebra for the *uniform* norm:

$$\|T\|_\infty = \sup_{\|\xi\| \leq 1} \|T\xi\|.$$

We moreover have $\|ST\|_\infty \leq \|S\|_\infty \|T\|_\infty$ for all $S, T \in \mathbf{B}(H)$. The algebra $\mathbf{B}(H)$ is naturally endowed with a $*$-operation called the *adjoint* operation defined as follows:

$$\langle T^*\xi, \eta \rangle = \langle \xi, T\eta \rangle, \forall \xi, \eta \in H.$$

We have $(T^*)^* = T$, $\|T^*\|_\infty = \|T\|_\infty$ and

$$\|T^*T\|_\infty = \|TT^*\|_\infty = \|T\|_\infty^2.$$

Thus, $\mathbf{B}(H)$ is a C*-algebra. We can define several *weaker* topologies on $\mathbf{B}(H)$ as well, in the following way.

Definition 1. Let H be a complex Hilbert space.

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- The *strong operator topology* (SOT) on $\mathbf{B}(H)$ is defined by the following family of open neighborhoods: for $S \in \mathbf{B}(H)$, $\varepsilon > 0$, $\xi_1, \dots, \xi_n \in H$, define
$$\mathcal{U}(S, \varepsilon, \xi_i) = \{T \in \mathbf{B}(H) : \|(T - S)\xi_i\| < \varepsilon, \forall 1 \leq i \leq n\}.$$
- The *weak operator topology* (WOT) on $\mathbf{B}(H)$ is defined by the following family of open neighborhoods: for $S \in \mathbf{B}(H)$, $\varepsilon > 0$, $\xi_1, \dots, \xi_n$, $\eta_1, \dots, \eta_n \in H$, define
$$\mathcal{V}(S, \varepsilon, \xi_i, \eta_i) = \{T \in \mathbf{B}(H) : |\langle (T - S)\xi_i, \eta_i\rangle| < \varepsilon, \forall 1 \leq i \leq n\}.$$

The uniform topology is stronger than the SOT which is itself stronger than the WOT.

Proposition 1. *Let $V \subset \mathbf{B}(H)$ be a weakly closed subspace and $\varphi : V \to \mathbf{C}$ a bounded linear functional. The following are equivalent.*

(1) *There exist $\xi_1, \dots, \xi_n, \eta_1, \dots, \eta_n \in H$ such that*
$$\varphi(T) = \sum_{i=1}^{n} \langle T\xi_i, \eta_i\rangle, \forall T \in V.$$
(2) *φ is strongly continuous.*
(3) *φ is weakly continuous.*

Moreover, for any nonempty convex subset $\mathcal{C} \subset \mathbf{B}(H)$, the strong operator closure and the weak operator closure of $\mathcal{C}$ coincide.

The following is the central definition of this introduction.

Definition 2. Let $M \subset \mathbf{B}(H)$ be a unital $*$-subalgebra. We say that M is a *von Neumann algebra* if M is weakly closed.

For a non-empty subset $\mathcal{S} \subset \mathbf{B}(H)$, define the *commutant* of $\mathcal{S}$ in $\mathbf{B}(H)$ by $\mathcal{S}' = \{T \in \mathbf{B}(H) : ST = TS, \forall S \in \mathcal{S}\}$. One can then define the double commutant by $\mathcal{S}'' = (\mathcal{S}')'$.

Theorem 1 (Von Neumann's Double Commutant Theorem)**.** *Let $M \subset \mathbf{B}(H)$ be a unital $*$-subalgebra. The following are equivalent:*

(1) *$M'' = M$.*
(2) *M is strongly closed.*
(3) *M is weakly closed.*

We will need to introduce more topologies to capture the intrinsic structure of a von Neumann algebra. We regard $\mathbf{B}(H) \subset \mathbf{B}(H \otimes \ell^2)$ via the embedding $T \mapsto T \otimes 1$

Definition 3. Let H be a complex Hilbert space.

- The *σ-strong operator topology* (σ-SOT) on $\mathbf{B}(H)$ is defined by restricting the SOT on $\mathbf{B}(H \otimes \ell^2)$ to $\mathbf{B}(H)$.

- The *σ-weak operator topology* (σ-WOT) on $\mathbf{B}(H)$ is defined by restricting the WOT on $\mathbf{B}(H \otimes \ell^2)$ to $\mathbf{B}(H)$.

Observe that on bounded subsets, SOT coincides with σ-SOT and WOT coincides with σ-WOT.

For von Neumann algebras M and N, we say that a bounded linear mapping $\phi : M \to N$ is *normal* if it is σ-weakly continuous. We then have the analogue of Proposition 1.

Proposition 2. *Let M be a von Neumann algebra and $\varphi : M \to \mathbf{C}$ be a bounded linear functional. The following are equivalent.*

(1) *There exist $(\xi_n)_{n\in\mathbf{N}}, (\eta_n)_{n\in\mathbf{N}} \in H \otimes \ell^2$ such that*

$$\varphi(T) = \sum_n \langle T\xi_n, \eta_n\rangle, \forall T \in M.$$

(2) *φ is σ-strongly continuous.*
(3) *φ is σ-weakly continuous, that is, normal*
(4) Monotone convergence*: For any bounded increasing net (x_i), we have $\varphi(\lim_i x_i) = \lim_i \varphi(x_i)$.*
(5) σ-additivity*: For any family of pairwise orthgonal projections (p_i), we have $\varphi(\sum_i p_i) = \sum_i \varphi(p_i)$.*

Moreover, for any nonempty convex subset $\mathcal{C} \subset \mathbf{B}(H)$, the σ-strong operator closure and the σ-weak operator closure of $\mathcal{C}$ coincide.

We denote by M_* the Banach space of all σ-weakly continuous bounded linear functionals on M. The mapping $\Phi : M \to (M_*)^*$ defined by $\Phi(x)(\varphi) = \varphi$ is surjective and isometric, that is,

$$\|x\|_\infty = \sup_{\varphi \in M_*, \|\varphi\| \leq 1} |\varphi(x)|.$$

Moreover, Φ is continuous when we endow M with the σ-WOT and $(M_*)^*$ with the weak-$*$ topology. We refer to M_* as the *predual* of M which is unique up to isometric isomorphism (Sakai). Thus, the σ-WOT is canonical and only depends on M. It also follows that any $*$-isomorphism between von Neumann algebras is necessarily normal.

We will denote by $\mathcal{U}(M)$ the group of unitary elements of M and by $\mathcal{Z}(M) = M' \cap M$ the center of M. We say that M is a *factor* if $\mathcal{Z}(M) = \mathbf{C}$.

Convention. All the von Neumann algebras that we consider are always assumed to have a separable predual and all the discrete groups that we consider are assumed to be countable.

Tracial von Neumann algebras. A von Neumann algebra M is said to be *tracial* if it is endowed with a faithful normal state τ which satisfies the *trace* relation:

$$\tau(xy) = \tau(yx), \forall x, y \in M.$$

Such a tracial state will be refered to as a *trace*. We will say that M is a II_1 factor if M is an infinite dimensional tracial von Neumann algebra and a factor.

Let (M, τ) be a tracial von Neumann algebra. We endow M with the following inner product

$$\langle x, y\rangle_\tau = \tau(y^* x), \forall x, y \in M.$$

Denote by $\mathrm{L}^2(M, \tau)$ or simply by $\mathrm{L}^2(M)$ the completion of M with respect to $\langle \cdot, \cdot\rangle_\tau$. The corresponding $\|\cdot\|_2$-norm on M is defined by $\|x\|_2 = \sqrt{\tau(x^*x)}$. Write $M \ni x \to \widehat{x} \in \mathrm{L}^2(M)$ for the canonical embedding. Note that the unit vector $\widehat{1}$ is *cyclic*, that is, $\widehat{M}$ is dense in $\mathrm{L}^2(M)$ and *separating*, that is, $x\widehat{1} = 0 \Rightarrow x = 0$ for all $x \in M$. For all $x, y \in M$,

$$\begin{aligned} \|xy\|_2^2 &= \tau(y^* x^* x y) \\ &\leq \tau(y^* \|x^* x\|_\infty y) \\ &\leq \|x\|_\infty^2 \|y\|_2^2, \end{aligned}$$

so that we can represent M in a *standard way* on $\mathrm{L}^2(M)$ by

$$\pi(x)\widehat{y} = \widehat{xy}, \forall x, y \in M.$$

This is the *GNS-representation*. Observe that $\pi : M \to \mathbf{B}(\mathrm{L}^2(M))$ is a normal $*$-representation and is isometric: $\|\pi(x)\|_\infty = \|x\|_\infty$ for all $x \in M$. Abusing notation, we identify $\pi(x)$ with $x \in M$ and regard $M \subset \mathbf{B}(\mathrm{L}^2(M))$. Define $J : \widehat{M} \ni \widehat{x} \mapsto \widehat{x^*} \in \mathrm{L}^2(M)$. For all $x, y \in M$, we have

$$\langle J\widehat{x}, J\widehat{y}\rangle = \langle \widehat{x^*}, \widehat{y^*}\rangle = \tau(yx^*) = \tau(x^*y) = \langle \widehat{y}, \widehat{x}\rangle.$$

Thus $J : \mathrm{L}^2(M) \to \mathrm{L}^2(M)$ is a conjugate linear unitary such that $J^2 = 1$.

Theorem 2. *We have $JMJ = M'$.*

Proof. We first prove $JMJ \subset M'$. Let $x, y, a \in M$. We have

$$JxJy\,\widehat{a} = Jx\widehat{a^*y^*} = \widehat{yax^*} = y\widehat{ax^*} = yJ\widehat{xa^*} = yJxJ\,\widehat{a}$$

so that $JxJy = yJxJ$.

Claim. The faithful normal state $x \mapsto \langle x\widehat{1}, \widehat{1}\rangle$ is a trace on M'.

Let $x \in M'$. We first show that $Jx\widehat{1} = x^*\widehat{1}$. Indeed, for every $a \in M$, we have

$$\begin{aligned} \langle Jx\widehat{1}, a\widehat{1}\rangle &= \langle Ja\widehat{1}, x\widehat{1}\rangle = \langle x^* a^* \widehat{1}, \widehat{1}\rangle \\ &= \langle a^* x^* \widehat{1}, \widehat{1}\rangle = \langle x^*\widehat{1}, a\widehat{1}\rangle. \end{aligned}$$

Let now $x, y \in M'$. We have

$$\begin{aligned}\langle xy\widehat{1}, \widehat{1}\rangle &= \langle y\widehat{1}, x^*\widehat{1}\rangle = \langle y\widehat{1}, Jx\widehat{1}\rangle = \langle x\widehat{1}, Jy\widehat{1}\rangle \\ &= \langle x\widehat{1}, y^*\widehat{1}\rangle = \langle yx\widehat{1}, \widehat{1}\rangle.\end{aligned}$$

Denote the faithful normal trace $x \mapsto \langle x\widehat{1}, \widehat{1}\rangle$ on M' by τ'. Define the canonical antiunitary K on $\mathrm{L}^2(M', \tau') = \overline{M'\widehat{1}} = \mathrm{L}^2(M)$ by $Kx\widehat{1} = x^*\widehat{1}$, $\forall x \in M'$. The first part of the proof yields $KM'K \subset M'' = M$. Since K and J coincide on $M'\widehat{1}$, which is dense in $\mathrm{L}^2(M)$, it follows that $K = J$. Therefore, we have $JM'J \subset M$ and so $JMJ = M'$. □

For all $x \in M$, put $\|x\|_1 = \tau(|x|)$.

Lemma 1. *The following hold:*

(1) *For all $x, y \in M$, we have $|\tau(xy)| \leq \|x\|_1 \|y\|_\infty$.*
(2) *For all $x, y \in M$, we have $\|x + y\|_1 \leq \|x\|_1 + \|y\|_1$.*

Proof. (1) Let $x, y \in M$ and write $x = u|x|$ (resp. $y = v|y|$) for the polar decomposition of x (resp. y) in M. By Cauchy Schwarz inequality and using the trace relation, we have

$$\begin{aligned}|\tau(xy)| &= |\tau(u|x|^{1/2}|x|^{1/2}\, v|y|^{1/2}|y|^{1/2})| \\ &= |\tau(|x|^{1/2}v|y|^{1/2}\, |y|^{1/2}u|x|^{1/2})| \\ &\leq \tau(|x|^{1/2}v|y|v^*|x|^{1/2})^{1/2}\, \tau(|y|^{1/2}u|x|u^*|y|^{1/2})^{1/2} \\ &= \tau(|x|^{1/2}|y^*||x|^{1/2})^{1/2}\, \tau(|y|^{1/2}|x^*||y|^{1/2})^{1/2} \\ &= \tau(|x|^{1/2}|y^*||x|^{1/2})^{1/2}\, \tau(|x^*|^{1/2}|y||x^*|^{1/2})^{1/2} \\ &\leq \|y\|_\infty^{1/2}\tau(|x|)^{1/2}\, \|y\|_\infty^{1/2}\tau(|x^*|)^{1/2} \\ &= \|y\|_\infty \tau(|x|).\end{aligned}$$

In particular, we obtain $|\tau(x)| \leq \tau(|x|)$ for all $x \in M$.

(2) Let $x, y \in M$ and write $x + y = u|x + y|$ for the polar decomposition of $x + y$ in M. Using (1), we have

$$\tau(|x + y|) = |\tau(u^*(x + y))| \leq |\tau(u^*x)| + |\tau(u^*y)| \leq \tau(|x|) + \tau(|y|). \quad \square$$

Define $\mathrm{L}^1(M, \tau)$ the completion of M with respect to the L^1-norm $\|\cdot\|_1$. The previous lemma allows moreover to define a linear mapping $\Psi : M \to M_*$ by the formula

$$\Psi(x)(y) = \tau(xy), \forall x, y \in M.$$

Indeed by the lemma, we know that $\|\Psi(x)\|_{M_*} \leq \|x\|_1$. Moreover, if $x = u^*|x|$ is the polar decomposition of x in M, we have $\Psi(x)(u^*) = \tau(xu^*) = \tau(u^*x) = \tau(|x|)$. Therefore $\|\Psi(x)\|_{M_*} = \|x\|_1$ for all $x \in M$ and so $\Psi : M \to M_*$ is a linear isometric embedding.

By density, we can then extend $\Psi : \mathrm{L}^1(M, \tau) \to M_*$ to a linear isometric embedding. One can also prove that the mapping Ψ is surjective. Therefore, we have the following.

Theorem 3. *Let (M, τ) be a tracial von Neumann algebra. Then $\Psi : \mathrm{L}^1(M, \tau) \to M_*$ as defined above is an isometric and surjective linear mapping.*

We will write $\tau(by) = \Psi(b)(y)$ for all $b \in \mathrm{L}^1(M, \tau)$ and all $y \in M$. From now on, we will always identify the predual of M with the Banach space $\mathrm{L}^1(M, \tau)$.

Abelian von Neumann algebras. Let (X, μ) be a standard probability space. Define the $*$-representation $\pi : \mathrm{L}^\infty(X, \mu) \to \mathbf{B}(\mathrm{L}^2(X, \mu))$ given by multiplication: $(\pi(f)\xi)(x) = f(x)\xi(x)$ for all $f \in \mathrm{L}^\infty(x, \mu)$ and all $\xi \in \mathrm{L}^2(X, \mu)$. Since π is a C*-algebraic isomorphism, we will identify $f \in \mathrm{L}^\infty(X, \mu)$ with its image $\pi(f) \in \mathbf{B}(\mathrm{L}^2(X, \mu))$. From now on, we will simply denote $\mathrm{L}^\infty(X, \mu)$ by $\mathrm{L}^\infty(X)$.

Proposition 3. *We have $\mathrm{L}^\infty(X)' \cap \mathbf{B}(\mathrm{L}^2(X, \mu)) = \mathrm{L}^\infty(X)$, that is, $\mathrm{L}^\infty(X)$ is maximal abelian in $\mathbf{B}(\mathrm{L}^2(X, \mu))$. In particular, $\mathrm{L}^\infty(X)$ is a von Neumann algebra.*

Proof. Let $T \in \mathrm{L}^\infty(X)' \cap \mathbf{B}(\mathrm{L}^2(X, \mu))$ and denote $f = T\mathbf{1}_X \in \mathrm{L}^2(X, \mu)$. For all $\xi \in \mathrm{L}^\infty(X) \subset \mathrm{L}^2(X, \mu)$, we have

$$T\xi = T\xi\, \mathbf{1}_X = \xi T\, \mathbf{1}_X = \xi f = f\xi.$$

For every $n \geq 1$, put $\mathcal{U}_n = \{x \in X : |f(x)| \geq \|T\|_\infty + \frac{1}{n}\}$. We have

$$(\|T\|_\infty + \frac{1}{n})\, \mu(\mathcal{U}_n)^{1/2} \leq \|f\mathbf{1}_{\mathcal{U}_n}\|_2 = \|T\mathbf{1}_{\mathcal{U}_n}\|_2 \leq \|T\|_\infty\, \mu(\mathcal{U}_n)^{1/2},$$

hence $\mu(\mathcal{U}_n) = 0$ for every $n \geq 1$. This implies that $\|f\|_\infty \leq \|T\|_\infty$ and so $T = f$. ☐

The von Neumann algebra $M = \mathrm{L}^\infty(X)$ comes equipped with the faithful normal trace τ_μ given by integration against the probability measure μ,

$$\tau_\mu(f) = \int_X f \mathrm{d}\mu, \forall f \in \mathrm{L}^\infty(X).$$

Using the Spectral Theorem, one can show that any abelian von Neumann algebra A with separable predual arises from a standard probability space, that is, there exists a standard probability space (X, μ) such that $A \cong \mathrm{L}^\infty(X)$.

Group von Neumann algebras. Let Γ be a countable discrete group. The *left* regular representation $\lambda : \Gamma \to \mathcal{U}(\ell^2(\Gamma))$ is defined by $\lambda_s \delta_t = \delta_{st}$.

Definition 4 (Group von Neumann algebra)**.** The von Neumann algebra $\mathrm{L}(\Gamma)$ is defined as the weak closure of the linear span of $\{\lambda_s : s \in \Gamma\}$.

Likewise, we can define the *right* regular representation $\rho : \Gamma \to \mathcal{U}(\ell^2(\Gamma))$ by $\rho_s \delta_t = \delta_{ts^{-1}}$. The *right* von Neumann algebra $\mathrm{R}(\Gamma)$ is defined as the weak closure of the linear span of $\{\rho_s : s \in \Gamma\}$. We obviously have $\mathrm{L}(\Gamma) \subset \mathrm{R}(\Gamma)'$.

Proposition 4. *The vector state* $\tau : \mathrm{L}(\Gamma) \to \mathbf{C}$ *defined by* $\tau(x) = \langle x\delta_e, \delta_e\rangle$ *is a faithful normal trace. Moreover* $\mathrm{L}(\Gamma) = \mathrm{R}(\Gamma)'$.

Proof. It is clear that τ is normal. We moreover have

$$\tau(u_s u_t) = \tau(u_{st}) = \delta_{st,e} = \delta_{ts,e} = \tau(u_{ts}) = \tau(u_t u_s).$$

It follows that τ is a trace on $\mathrm{L}(\Gamma)$. Assume now that $\tau(x^*x) = 0$, that is, $x\delta_e = 0$ for $x \in \mathrm{L}(\Gamma)$. For all $t \in \Gamma$, we have $x\delta_t = x\rho_{t^{-1}}\delta_e = \rho_{t^{-1}}x\delta_e = 0$. Therefore $x = 0$. Hence τ is faithful.

We can identify $\ell^2(\Gamma)$ with $\mathrm{L}^2(\mathrm{L}(\Gamma))$ via the unitary mapping $\delta_g \mapsto u_g$. Under this identification, we have $J\delta_t = \delta_{t^{-1}}$. An easy calculation shows that for all $s, t \in \Gamma$, we have

$$J\lambda_s J\, \delta_t = J\lambda_s \delta_{t^{-1}} = J\delta_{st^{-1}} = \delta_{ts^{-1}} = \rho_s\, \delta_t.$$

Therefore, $J\lambda_s J = \rho_s$ for all $s \in \Gamma$. It follows that $\mathrm{L}(\Gamma)' = J\mathrm{L}(\Gamma)J = \mathrm{R}(\Gamma)$ and thus $\mathrm{L}(\Gamma) = \mathrm{R}(\Gamma)'$. ∎

Let $x \in \mathrm{L}(\Gamma)$ and write $x\delta_e = \sum_{s\in\Gamma} x_s \delta_s \in \ell^2(\Gamma)$ with $x_s = \langle x\delta_e, \delta_s\rangle = \tau(x\lambda_s^*)$ for all $s \in \Gamma$. As we have seen, the family $(x_s)_{s\in\Gamma}$ completely determines $x \in \Gamma$. We shall denote by $x = \sum_{s\in\Gamma} x_s\lambda_s$ the *Fourier expansion* of $x \in \mathrm{L}(\Gamma)$.

Warning. The above sum $\sum_{s\in\Gamma} x_s\lambda_s$ **does not converge** in general for any of the topologies on $\mathbf{B}(\ell^2(\Gamma))$. However, the net of finite sums $(x_{\mathcal{F}})_{\mathcal{F}}$ defined by $x_{\mathcal{F}} = \sum_{s\in\mathcal{F}} x_s\lambda_s$ for $\mathcal{F} \subset \Gamma$ a finite subset does converge for the $\|\cdot\|_2$-norm. Indeed since $(x_s) \in \ell^2(\Gamma)$, for any $\varepsilon > 0$, there exists $\mathcal{F}_0 \subset \Gamma$ finite subset such that $\sum_{s\in\Gamma\setminus\mathcal{F}_0} |x_s|^2 \leq \varepsilon^2$. Thus, for every finite subset $\mathcal{F} \subset \Gamma$ such that $\mathcal{F}_0 \subset \mathcal{F}$, we have $\|x - x_{\mathcal{F}}\|_2^2 = \sum_{s\in\Gamma\setminus\mathcal{F}} |x_s|^2 \leq \varepsilon^2$.

The notation $x = \sum_{s\in\Gamma} x_s\lambda_s$ behaves well with respect to taking the adjoint and multiplication.

Proposition 5. *Let* $x = \sum_{s\in\Gamma} x_s\lambda_s$ *(resp.* $y = \sum_{t\in\Gamma} y_t\lambda_t$*) be the Fourier expansion of* $x \in \mathrm{L}(\Gamma)$ *(resp.* $y \in \mathrm{L}(\Gamma)$*). Then we have*

- $x^* = \sum_{s\in\Gamma} \overline{x_{s^{-1}}}\lambda_s$.
- $xy = \sum_{t\in\Gamma} \left(\sum_{t\in\Gamma} x_s y_{s^{-1}t}\right)\lambda_t$*, with* $\sum_{s\in\Gamma} x_s y_{s^{-1}t} \in \mathbf{C}$ *for all* $t \in \Gamma$*, by Cauchy-Schwarz inequality.*

Proof. For the first item, observe that

$$(x^*)_s = \tau(x^*\lambda_s^*) = \overline{\tau(\lambda_s x)} = \overline{\tau(x\lambda_{s^{-1}}^*)} = \overline{x_{s^{-1}}}.$$

For the second item, observe that using Cauchy-Schwarz inequality, we have

$$(xy)_t = \tau(xy\lambda_t^*) = \sum_{s\in\Gamma} x_s \tau(\lambda_s y \lambda_t^*) = \sum_{s\in\Gamma} x_s \tau(y\lambda_{s^{-1}t}^*) = \sum_{s\in\Gamma} x_s y_{s^{-1}t}. \quad \square$$

Thanks to the Fourier expansion, we can compute the center $\mathcal{Z}(\mathrm{L}(\Gamma))$ of the group von Neumann algebra. We say that Γ is icc (infinite conjugacy classes) if for every $s \in \Gamma \setminus \{e\}$, the conjugacy class $\{tst^{-1} : t \in \Gamma\}$ is infinite.

Proposition 6. *We have $x = \sum_{s\in\Gamma} x_s\lambda_s \in \mathcal{Z}(\mathrm{L}(\Gamma))$ if and only if $x_{tst^{-1}} = x_s$ for all $s, t \in \Gamma$. In particular, $\mathrm{L}(\Gamma)$ is a factor if and only if Γ is icc. Thus, $\mathrm{L}(\Gamma)$ is a II_1 factor whenever Γ is infinite and icc.*

Proof. We have

$$\begin{aligned} x = \sum_{s\in\Gamma} x_s\lambda_s \in \mathcal{Z}(\mathrm{L}(\Gamma)) &\Leftrightarrow \lambda_t^* x \lambda_t = x, \forall s \in \Gamma \\ &\Leftrightarrow x_{tst^{-1}} = x_s, \forall s, t \in \Gamma. \end{aligned}$$

If Γ is icc and $x \in \mathcal{Z}(\mathrm{L}(\Gamma))$, since $(x_{tst^{-1}})_t \in \ell^2(\Gamma)$, for all $s \in \Gamma$, it follows that $x_s = 0$ for all $s \in \Gamma \setminus \{e\}$. Hence $\mathcal{Z}(\mathrm{L}(\Gamma)) = \mathbf{C}$.

If Γ is not icc, then $F = \{tst^{-1} : t \in \Gamma\}$ is finite for some $s \in \Gamma \setminus \{e\}$. Then $\sum_{h\in F} \lambda_h \in \mathcal{Z}(\mathrm{L}(\Gamma)) \setminus \mathbf{C}$. $\square$

Example 1. Here are a few examples of icc groups: the subgroup $S_\infty < S(\mathbf{N})$ of finitely supported permutations; the free groups $\mathbf{F}_n$ for $n \geq 2$; the lattices $\mathrm{PSL}(n, \mathbf{Z})$ for $n \geq 2$.

Hence Proposition 6 provides many examples of II_1 factors arising from countable discrete groups.

Exercise 1. Let $T = [T_{st}]_{s,t\in\Gamma} \in \mathbf{B}(\ell^2(\Gamma))$, with $T_{st} = \langle T\delta_t, \delta_s\rangle$. Show that $T \in L(\Gamma)$ if and only if T is *constant down the diagonals*, that is, $T_{st} = T_{gh}$ whenever $st^{-1} = gh^{-1}$.

Example 2. Assume that Γ is abelian. Then the dual $\widehat{\Gamma}$ is a compact second countable abelian group. Write $\mathcal{F} : \ell^2(\Gamma) \to \mathrm{L}^2(\widehat{\Gamma}, \mathrm{Haar})$ for the Fourier transform which is defined by $\mathcal{F}(\delta_s)(\chi) = \langle s, \chi\rangle$. Observe that $\mathcal{F}$ is a unitary operator. We then get

$$\mathrm{L}^\infty(\widehat{\Gamma}) = \mathcal{F}\mathrm{L}(\Gamma)\mathcal{F}^*.$$

Lecture 2

In the second lecture, we study Murray-von Neumann's group measure space construction and we introduce the central concept of Cartan subalgebra.

Murray-von Neumann's group measure space construction. Let $\Gamma \curvearrowright (X, \mu)$ be a probability measure preserving (pmp) action. Define the action $\sigma : \Gamma \curvearrowright \mathrm{L}^\infty(X)$ by $(\sigma_s(F))(x) = F(s^{-1}x)$, $\forall F \in \mathrm{L}^\infty(X)$. This action extends to a unitary representation $\sigma : \Gamma \to \mathcal{U}(\mathrm{L}^2(X))$. Put $H = \mathrm{L}^2(X) \otimes \ell^2(\Gamma)$. Put $u_s = \sigma_s \otimes \lambda_s$ for all $s \in \Gamma$. Observe that by Fell's absorption principle, the representation $\Gamma \to \mathcal{U}(H) : s \mapsto u_s$ is unitarily conjugate to a multiple of the left regular representation. We will identify $F \in \mathrm{L}^\infty(X)$ with $F \otimes 1 \in \mathrm{L}^\infty(X) \otimes 1$.

We have the following *covariance* relation:

$$u_s F u_s^* = \sigma_s(F), \forall F \in \mathrm{L}^\infty(X), \forall s \in \Gamma.$$

Definition 5 (Murray, von Neumann)**.** The *group measure space construction* $\mathrm{L}^\infty(X) \rtimes \Gamma$ is defined as the weak closure of the linear span of $\{F u_s : F \in \mathrm{L}^\infty(X), s \in \Gamma\}$.

Put $M = \mathrm{L}^\infty(X) \rtimes \Gamma$. Define the unital faithful $*$-representation $\pi : \mathrm{L}^\infty(X) \to \mathbf{B}(H)$ by $\pi(F)(\xi \otimes \delta_t) = \sigma_t(F)\xi \otimes \delta_t$. Denote by N the von Neumann algebra acting on H generated by $\pi(\mathrm{L}^\infty(X))$ and $(1 \otimes \rho)(\Gamma)$. It is straightforward to check that $M \subset N'$.

Proposition 7. *The vector state $\tau : M \to \mathbf{C}$ defined by $\tau(x) = \langle x(\mathbf{1}_X \otimes \delta_e), \mathbf{1}_X \otimes \delta_e\rangle$ is a faithful normal trace. Moreover we have $M = N'$.*

Proof. It is clear that τ is normal. We moreover have

$$\begin{aligned}
\tau(F u_s\, G u_t) &= \tau(F \sigma_s(G) u_{st}) \\
&= \delta_{st,e} \int_X F(x) G(s^{-1}x) \mathrm{d}\mu(x) \\
&= \delta_{st,e} \int_X F(sx) G(x) \mathrm{d}\mu(x) \\
&= \delta_{ts,e} \int_X G(x) F(t^{-1}x) \mathrm{d}\mu(x) \\
&= \tau(G \sigma_t(F) u_{ts}) \\
&= \tau(G u_t\, F u_s).
\end{aligned}$$

It follows that τ is a trace on M. Assume that $\tau(b^*b) = 0$, that is, $b(\mathbf{1}_X \otimes \delta_e) = 0$. For all $s \in \Gamma$ and all $F \in \mathrm{L}^\infty(X)$, we have

$$\begin{aligned}
b\,(F \otimes \delta_t) &= b\,\pi(\sigma_{t^{-1}}(F))(1 \otimes \rho_{t^{-1}})(\mathbf{1}_X \otimes \delta_e) \\
&= \pi(\sigma_{t^{-1}}(F))(1 \otimes \rho_{t^{-1}})\, b(\mathbf{1}_X \otimes \delta_e) = 0.
\end{aligned}$$

It follows that $b = 0$. Hence τ is faithful.

We will identify $\mathrm{L}^2(M)$ with $\mathrm{L}^2(X) \otimes \ell^2(\Gamma)$ via the unitary mapping $F u_s \mapsto F \otimes \delta_s$. Under this identification, the conjugation $J : \mathrm{L}^2(M) \to \mathrm{L}^2(M)$ is defined by $J(\xi \otimes \delta_s) = \sigma_{s^{-1}}(\xi^*) \otimes \delta_{s^{-1}}$. For all $F \in \mathrm{L}^\infty(X)$ and all $s \in \Gamma$, we have

$$J(\sigma_s \otimes \lambda_s)J = 1 \otimes \rho_s$$
$$J(F \otimes 1)J = \pi(F)^*.$$

Therefore, we get $M = N'$. □

Observe that when the probability space $X = \{\bullet\}$ is a point, then the group von Neumann algebra and the group measure space construction coincide, that is, $\mathrm{L}^\infty(X) \rtimes \Gamma = \mathrm{L}(\Gamma)$.

Let $\Gamma \curvearrowright (X, \mu)$ be a pmp action. Put $A = \mathrm{L}^\infty(X)$ and $M = \mathrm{L}^\infty(X) \rtimes \Gamma$. Recall that we may regard $M \subset \mathbf{B}(\mathrm{L}^2(X) \otimes \ell^2(\Gamma))$. Let $\omega \in \mathbf{B}(\ell^2(\Gamma))$ be the normal vector state defined by $\omega = \langle \cdot \delta_e, \delta_e \rangle$. For all $T \in \mathbf{B}(\mathrm{L}^2(X) \otimes \ell^2(\Gamma))$, denote by $(\mathrm{id} \otimes \omega)(T)$ the unique bounded operator on $\mathrm{L}^2(X)$ which satisfies

$$\langle (\mathrm{id} \otimes \omega)(T)\xi, \eta \rangle = \langle T(\xi \otimes \delta_e), \eta \otimes \delta_e \rangle$$

for all $\xi, \eta \in \mathrm{L}^2(X)$. One checks that the map $\mathbf{B}(\mathrm{L}^2(X) \otimes \ell^2(\Gamma)) \ni T \mapsto (\mathrm{id} \otimes \omega)(T) \in \mathbf{B}(\mathrm{L}^2(X))$ is bounded linear positive and normal.

Proposition 8 (Conditional expectation)**.** *Define $E(b) = (\mathrm{id} \otimes \omega)(b)$ for all $b \in M$. We have $E(au_t) = \delta_{t,e} a$ for all $a \in A$ and all $t \in \Gamma$. Therefore E takes values in $A \subset \mathbf{B}(\mathrm{L}^2(X))$. Moreover $E : M \to A$ satisfies the following properties:*

- *$E : M \to A$ is faithful, that is, for all $b \in M$ such that $E(b^*b) = 0$ then $b = 0$.*
- *$E(a_1 b a_2) = a_1 E(b) a_2$ for all $a_1, a_2 \in A$ and all $b \in M$.*
- *$\tau(E(b)) = \tau(b)$ for all $b \in M$.*

Proof. For all $\xi, \eta \in \mathrm{L}^2(X, \mu)$, all $t \in \Gamma$ and all $a \in A$, we have

$$\langle E(au_t)\xi, \eta \rangle = \langle (a\sigma_t \otimes \lambda_t)(\xi \otimes \delta_e), \eta \otimes \delta_e \rangle = \langle a\sigma_t \xi, \eta \rangle \langle \lambda_t \delta_e, \delta_e \rangle = \delta_{t,e} \langle a\xi, \eta \rangle.$$

Therefore, we have $E(au_t) = \delta_{t,e} a$. The rest is routine to check and left to the reader. □

The map $E : M \to A$ is called the *conditional expectation* from M onto A. We will encounter a much more general phenomenon in the third lecture. From now on, we will denote it by $E_A : M \to A$. It is easy to see that $E_A : M \to A$ is the unique trace preserving conditional expectation. We will denote by $e_A : \mathrm{L}^2(M) \to \mathrm{L}^2(A)$ the orthogonal projection. It moreover satisfies

$$e_A(\widehat{a}) = \widehat{E_A(a)} \text{ and } e_A a e_A = E_A(a) e_A, \forall a \in M.$$

Proposition 9 (Fourier expansion)**.** *Let $\Gamma \curvearrowright (X, \mu)$ be a pmp action. Let $A = \mathrm{L}^\infty(X)$ and $M = \mathrm{L}^\infty(X) \rtimes \Gamma$. Every $a \in M$ has a unique* Fourier expansion *of the form $a = \sum_{s \in \Gamma} a_s u_s$ with $a_s = E_A(a u_s^*)$. The convergence holds for the $\|\cdot\|_2$-norm. Moreover, we have the following:*

- $a^* = \sum_{s\in\Gamma} \sigma_{s^{-1}}(a_s^*) u_s$.
- $\|a\|_2^2 = \sum_{s\in\Gamma} \|a_s\|_2^2$.
- $ab = \sum_{t\in\Gamma} \left(\sum_{s\in\Gamma} a_s \sigma_s(b_{s^{-1}t})\right) u_t$.

Proof. Define the unitary mapping $U : \mathrm{L}^2(M) \to \mathrm{L}^2(X) \otimes \ell^2(\Gamma)$ by the formula $U(au_s) = a \otimes \delta_s$. Then $U\widehat{1}U^* = \mathbf{1}_X \otimes \delta_e$ is a cyclic separating vector for M represented on the Hilbert space $\mathrm{L}^2(X) \otimes \ell^2(\Gamma)$. We identify $\mathrm{L}^2(M)$ with $\mathrm{L}^2(X) \otimes \ell^2(\Gamma)$. Under this identification e_A is the orthogonal projection $\mathrm{L}^2(X) \otimes \ell^2(\Gamma) \to \mathrm{L}^2(X) \otimes \mathbf{C}\delta_e$. Moreover $u_s e_A u_s^*$ is the orthogonal projection $\mathrm{L}^2(X) \otimes \ell^2(\Gamma) \to \mathrm{L}^2(X) \otimes \mathbf{C}\delta_s$ and thus $\sum_{s\in\Gamma} u_s e_A u_s^* = 1$. Let $a \in M$. Regarding $a(\mathbf{1}_X \otimes \delta_e) \in \mathrm{L}^2(X) \otimes \ell^2(\Gamma)$, we know that there exists $a_s \in \mathrm{L}^2(X)$ such that

$$a(\mathbf{1}_X \otimes \delta_e) = \sum_{s\in\Gamma} a_s \otimes \delta_s \text{ and } \|a\|_2^2 = \sum_{s\in\Gamma} \|a_s\|_2^2.$$

Then we have

$$\begin{aligned} a_s \otimes \delta_s &= u_s e_A u_s^* a(\mathbf{1}_X \otimes \delta_e) \\ &= u_s e_A u_s^* a e_A(\mathbf{1}_X \otimes \delta_e) \\ &= u_s E_A(u_s^* a)(\mathbf{1}_X \otimes \delta_e) \\ &= E_A(a u_s^*) \otimes \delta_s. \end{aligned}$$

It follows that $a_s = E_A(au_s^*)$. Therefore, we have $a = \sum_{s\in\Gamma} E_A(au_s^*)u_s$ and the convergence holds for the $\|\cdot\|_2$-norm. Moreover, $\|a\|_2^2 = \sum_{s\in\Gamma} \|E_A(au_s^*)\|_2^2$. The rest of the proof is left to the reader. □

Warning. Like in the group case, the sum $a = \sum_{s\in\Gamma} a_s u_s$ **does not converge** in general for any of the operator topologies on $\mathbf{B}(\mathrm{L}^2(X) \otimes \ell^2(\Gamma))$.

Definition 6. Let $\Gamma \curvearrowright (X, \mu)$ be a pmp action.

- We say that the action is *(essentially) free* if $\mu(\{x \in X : sx = x\}) = 0$ for all $s \in \Gamma \setminus \{e\}$.
- We say that the action is *ergodic* if every Γ-invariant measurable subset $U \subset X$ has measure 0 or 1.

Lemma 2. *Let $\Gamma \curvearrowright (X, \mu)$ be a pmp action and denote by $\sigma : \Gamma \to \mathrm{L}^2(X)^0$ the corresponding Koopman representation where $\mathrm{L}^2(X)^0 = \mathrm{L}^2(X) \ominus \mathbf{C}\mathbf{1}_X$. The following are equivalent:*

1. *The action $\Gamma \curvearrowright (X, \mu)$ is ergodic.*
2. *The Koopman representation $\sigma \to \mathcal{U}(\mathrm{L}^2(X)^0)$ has no nonzero invariant vectors.*

Proof. (1) $\Rightarrow$ (2) Let $\xi \in \mathrm{L}^2(X)^0$ such that $\sigma_s(\xi) = \xi$ for all $s \in \Gamma$. For every non-negative real number y, define $U_y = \{x \in X : |\xi(x)|^2 \geq y\}$. It follows that U_y is Γ-invariant for all $y \geq 0$ and thus $\mu(U_y) = 0, 1$ by ergodicity. Since the fonction $y \mapsto \mu(U_y)$ is decreasing and since $\xi \in \mathrm{L}^2(X)$, there exists

y_0 such that $\mu(U_y) = 0$ for all $y \geq y_0$. Therefore $|\xi(x)|^2 = y_0$ for almost every $x \in X$. Since $\xi \in \mathrm{L}^2(X)^0$, we get $y_0 = 0$ and so $\xi = 0$.

(2) $\Rightarrow$ (1) Let $U \subset X$ be a Γ-invariant measurable subset. Put $\xi = \mathbf{1}_U - \mu(U)\mathbf{1}_X \in \mathrm{L}^2(X)^0$. Since $\sigma_s(\xi) = \xi$ for all $s \in \Gamma$, we get $\xi = 0$ and so $\mathbf{1}_U = \mu(U)\mathbf{1}_X$. Hence $\mu(U) = 0, 1$. $\square$

Example 3. Here are a few examples of pmp free ergodic actions $\Gamma \curvearrowright (X, \mu)$.

1. **Bernoulli actions.** Let Γ be an infinite group and (Y, η) a non-trivial probability space, that is, η is not a Dirac point mass. Put $(X, \mu) = (Y^\Gamma, \nu^{\otimes \Gamma})$. Consider the Bernoulli action $\Gamma \curvearrowright Y^\Gamma$ defined by
$$s \cdot (y_t)_{t \in \Gamma} = (y_{s^{-1}t})_{t \in \Gamma}.$$
Then the Bernoulli action is pmp free and mixing, so in particular ergodic.
2. **Profinite actions.** Let Γ be an infinite residually finite group together with a decreasing chain of finite index normal subgroups $\Gamma_n \lhd \Gamma$ such that $\Gamma_0 = \Gamma$ and $\cap_{n \in \mathbf{N}} \Gamma_n = \{e\}$. Then for all $n \geq 1$, the action $\Gamma \curvearrowright (\Gamma/\Gamma_n, \mu_n)$ is transitive and preserves the normalized counting measure μ_n. Consider the profinite action defined as the projective limit
$$\Gamma \curvearrowright (\mathbf{G}, \mu) = \varprojlim \Gamma \curvearrowright (\Gamma/\Gamma_n, \mu_n).$$
Then Γ sits as a dense subgroup of the compact group $\mathbf{G}$ which is the profinite completion of Γ with respect to the decreasing chain $(\Gamma_n)_{n \in \mathbf{N}}$. Observe that μ is the unique Haar probability measure on $\mathbf{G}$. The profinite action is pmp free and ergodic.
3. **Actions on tori.** Let $n \geq 2$. Consider the action $\mathrm{SL}(n, \mathbf{Z}) \curvearrowright (\mathbf{T}^n, \lambda_n)$ where $\mathbf{T}^n = \mathbf{R}^n/\mathbf{Z}^n$ is the n-torus and λ_n is the unique Haar probability measure. This action is pmp free and ergodic.

We always assume that (X, μ) is a standard probability space. In particular, X is *countably separated* in the sense that there exists a sequence of Borel subsets $V_n \subset X$ such that $\bigcup_n V_n = X$, $\mu(V_n) > 0$ for all $n \in \mathbf{N}$ and with the property that whenever $x, y \in X$ and $x \neq y$, there exists $n \in \mathbf{N}$ for which $x \in V_n$ and $y \notin V_n$.

Proposition 10. *Let $\Gamma \curvearrowright (X, \mu)$ be a pmp action. Put $A = \mathrm{L}^\infty(X)$ and $M = \mathrm{L}^\infty(X) \rtimes \Gamma$.*

1. *The action is free if and only if $A \subset M$ is maximal abelian, that is, $A' \cap M = A$.*
2. *Under the assumption that the action is free, the action is ergodic if and only if M is a factor.*

Proof. (1) Assume that the action is free. Let $b \in A' \cap M$ and write $b = \sum_{s\in\Gamma} b_s u_s$ for its Fourier expansion. Then for all $a \in A$ and all $s \in \Gamma$, we have $ab_s = \sigma_s(a)b_s$. Fix $s \in \Gamma \setminus \{e\}$ and put $U_s = \{x \in X : b_s(x) \neq 0, sx \neq x\}$. We have $\mathbf{1}_{U_s} a = \mathbf{1}_{U_s}\sigma_s(a)$ for all $a \in A$.

By assumption, we have $U_s = U_s \cap (\bigcup_n V_n \cap s(V_n)^c)$. So, if $\mu(U_s) > 0$, there exists $n \in \mathbf{N}$ such that $\mu(U_s \cap V_n \cap s(V_n)^c) > 0$. With $a = \mathbf{1}_{V_n}$, we get $\mathbf{1}_{U_s\cap V_n} = \mathbf{1}_{U_s\cap s(V_n)}$ and thus $\mathbf{1}_{U_s\cap V_n\cap s(V_n)^c} = 0$, which is a contradiction. Therefore, $\mu(U_s) = 0$. Since the action is moreover free, we get $b_s = 0$. This implies that $b \in A$.

Conversely, assume that $A' \cap M = A$. For all $s \in \Gamma \setminus \{e\}$, put $a_s = \mathbf{1}_{\{x\in X : sx=x\}}$. We have $a_s u_s \in A' \cap M = A$. Hence $a_s u_s = E_A(a_s u_s) = 0$ and so $a_s = 0$. Therefore $\mu(\{x \in X : sx = x\}) = 0$.

(2) Under the assumption that the action is free, we have $\mathcal{Z}(M) = M' \cap M = M' \cap A = A^\Gamma$. Therefore, the action is ergodic if and only if $\mathcal{Z}(M) = \mathbf{C}$. □

Cartan subalgebra.

Definition 7. Let (M, τ) be a tracial von Neumann algebra. We say that $A \subset M$ is a *Cartan subalgebra* if A is maximal abelian in M, that is, $A' \cap M = A$ and if the group $\mathcal{N}_M(A) = \{u \in \mathcal{U}(M) : uAu^* = A\}$ generates M.

When $\Gamma \curvearrowright (X, \mu)$ is a free pmp action, $\mathrm{L}^\infty(X) \subset \mathrm{L}^\infty(X) \rtimes \Gamma$ is a Cartan subalgebra by Proposition 10. We will be using the following ergodicity result.

Theorem 4. *Let A be a Cartan subalgebra in a II_1 factor M. Then for all projections $p, q \in A$ such that $\tau(p) = \tau(q)$, there exists $u \in \mathcal{N}_M(A)$ such that $upu^* = q$.*

Proof. Put $\mathcal{G} = \mathcal{N}_M(A)$. Let $p, q \in A$ be nonzero projections such that $\tau(p) = \tau(q)$. We start by proving the following.

Claim. Then there exists $u \in \mathcal{G}$ and nonzero projections $p_0, q_0 \in A$ such that $p_0 \leq p$, $q_0 \leq q$ and $up_0u^* = q_0$.

Since $\mathcal{G}''$ is a factor and $p \neq 0$, we have $\bigvee_{u\in\mathcal{G}} upu^* = 1$. Since $q \neq 0$, there exists $u \in \mathcal{G}$ such that $upu^* \wedge q \neq 0$. Letting $q_0 = upu^* \wedge q$ and $p_0 = u^*q_0u$, the claim is proven.

By Zorn's Lemma, choose a maximal family (p_i, q_i) with respect to inclusion of pairwise orthogonal projections $p_i \in Ap$ and pairwise orthogonal projections $q_i \in Aq$ such that for all i there exists $u_i \in \mathcal{G}$ which satisfies $q_i = u_i p_i u_i^*$. By maximality and using the Claim, we have that $\sum_i p_i = p$ and $\sum_i q_i = q$. Put $v = \sum_i u_i p_i \in M$ and observe that v is partial isometry in M such that $vApv^* = Aq$. Likewise, we get a partial isometry $w \in M$ such that $wAp^\perp w^* = Aq^\perp$. Letting $u = v + w$, we have $u \in \mathcal{G}$ and $upu^* = q$. □

Lecture 3

In the third lecture, we introduce several tools which are very useful in the structure and classification of II_1 factors. These include Connes's theory of bimodules, Jones's basic construction and Popa's intertwining techniques.

Connes's theory of bimodules. The discovery of the appropriate notion of representations for von Neumann algebras, as so-called *correspondences* or *bimodules*, is due to Connes. Whenever M is a von Neumann algebra, we denote by M^{op} the opposite von Neumann algebra.

Definition 8. Let M, N be tracial von Neumann algebras. A Hilbert space $\mathcal{H}$ is said to be an M-N-*bimodule* if it comes equipped with two commuting normal $*$-representations $\lambda : M \to \mathbf{B}(\mathcal{H})$ and $\rho : N^{\mathrm{op}} \to \mathbf{B}(\mathcal{H})$. We shall intuitively write

$$x\xi y = \lambda(x)\rho(y^{\mathrm{op}})\xi, \ \forall \xi \in \mathcal{H}, \forall x \in M, \forall y \in N.$$

We will sometimes denote by $\pi_{\mathcal{H}} : M \otimes_{\mathrm{alg}} N^{\mathrm{op}} \to \mathbf{B}(\mathcal{H})$ the unital $*$-representation associated with the M-N-bimodule structure on $\mathcal{H}$.

Example 4. The following are important examples of bimodules:

(1) The identity bimodule $\mathrm{L}^2(M)$ with $x\xi y = x\, Jy^* J\xi$.
(2) The coarse bimodule $\mathrm{L}^2(M) \otimes \mathrm{L}^2(N)$ with $x(\xi \otimes \eta)y = (x\xi) \otimes (\eta y)$.
(3) For any τ-preserving automorphism $\theta \in \mathrm{Aut}(M)$, we regard $\mathrm{L}^2_\theta(M)$ with the following M-M-bimodule structure: $x\xi y = x\xi\theta(y)$.

We will say that two M-N-bimodules ${}_M\mathcal{H}_N$ and ${}_M\mathcal{K}_N$ are *isomorphic* and write ${}_M\mathcal{H}_N \cong {}_M\mathcal{K}_N$ if there exists a unitary mapping $U : \mathcal{H} \to \mathcal{K}$ such that

$$U(x\xi y) = xU(\xi)y, \ \forall \xi \in \mathcal{H}, \forall x \in M, \forall y \in N.$$

We now describe *Connes's fusion tensor product* for Hilbert bimodules. Let M, N, P be any tracial von Neumann algebras, $\mathcal{H}$ any M-N-bimodule and $\mathcal{K}$ any N-P-bimodule. Denote by $\mathcal{H}_0 \subset \mathcal{H}$ the subspace of right N-bounded vectors, that is,

$$\mathcal{H}_0 = \{a \in \mathcal{H} : \exists c > 0, \forall y \in N, \|ay\| \leq c\|y\|_2\}.$$

Whenever $a \in \mathcal{H}_0$, we denote by $L_a : \mathrm{L}^2(N) \to \mathcal{H} : y \mapsto ay$ the corresponding bounded operator. Observe that for all $a, b \in \mathcal{H}_0$, we have

$$L_b^* L_a \in (JNJ)' \cap \mathbf{B}(\mathrm{L}^2(N)) = N.$$

Observe that $\mathcal{H}_0$ is dense in $\mathcal{H}$. Indeed, for every $\xi \in \mathcal{H}$, denote by $T_\xi \in \mathrm{L}^1(M, \tau)$ the unique element such that $\langle \xi y, \xi \rangle = \tau(T_\xi y)$ for all $y \in N$. Regarding T_ξ as a closed summable operator affiliated with N, we may take the spectral decomposition of T_ξ and find an increasing sequence of projection $e_n \in N$ such that $\xi e_n \in \mathcal{H}_0$ and $\xi e_n \to \xi$.

The separation/completion of $\mathcal{H}_0 \otimes_{\rm alg} \mathcal{K}$ with respect to the sesquilinear form

$$\langle a \otimes \xi, b \otimes \eta \rangle = \langle L_b^* L_a \xi, \eta \rangle_{\mathcal{K}}$$

is denoted by $\mathcal{H} \otimes_N \mathcal{K}$. The image of $a \otimes \eta \in \mathcal{H}_0 \otimes_{\rm alg} \mathcal{K}$ in $\mathcal{H} \otimes_N \mathcal{K}$ is simply denoted by $a \otimes_N \xi$. The M-P-bimodule structure on $\mathcal{H} \otimes_N \mathcal{K}$ is given by

$$x(a \otimes_N \xi)y = xa \otimes_N \xi y, \forall x \in M, \forall y \in P.$$

Exercise 2 (Associativity)**.** Let M, N, P, Q be any tracial von Neumann algebras and ${}_M\mathcal{K}_N$, ${}_N\mathcal{K}_P$, ${}_P\mathcal{L}_Q$ bimodules. Show that as M-Q-bimodules, we have

$${}_M((\mathcal{H} \otimes_N \mathcal{K}) \otimes_P \mathcal{L})_Q \cong {}_M(\mathcal{H} \otimes_N (\mathcal{K} \otimes_P \mathcal{L}))_Q.$$

Like for unitary group representations, we can define a notion of *weak containment* of Hilbert bimodules. Let M, N be any tracial von Neumann algebras and ${}_M\mathcal{H}_N, {}_M\mathcal{K}_N$ any bimodules. Consider the unital $*$-representations $\pi_{\mathcal{H}} : M \otimes_{\rm alg} N^{\rm op} \to \mathbf{B}(\mathcal{H})$ and $\pi_{\mathcal{K}} : M \otimes_{\rm alg} N^{\rm op} \to \mathbf{B}(\mathcal{K})$.

Definition 9 (Weak containment)**.** We say that $\mathcal{H}$ is *weakly contained* in $\mathcal{K}$ and write $\mathcal{H} \subset_{\rm weak} \mathcal{K}$ if $\|\pi_{\mathcal{H}}(T)\| \leq \|\pi_{\mathcal{K}}(T)\|$ for all $T \in M \otimes_{\rm alg} N^{\rm op}$.

Let $\pi : \Gamma \to \mathcal{U}(K_\pi)$ be a unitary representation of a countable discrete group Γ. Put $M = \mathrm{L}(\Gamma)$ and denote by $(u_s)_{s \in \Gamma}$ the canonical unitaries in M. Define on $\mathcal{H}(\pi) = K_\pi \otimes \ell^2(\Gamma)$ the following M-M-bimodule structure. For all $\xi \in K_\pi$ and all $s, t \in \Gamma$, define

$$u_s\,(\xi \otimes \delta_t) = \pi_s \xi \otimes \delta_{st}$$
$$(\xi \otimes \delta_t)\,u_s = \xi \otimes \delta_{ts}.$$

It is clear that the right multiplication extends to the whole von Neumann algebra M. Observe now that the unitary representations $\pi \otimes \lambda$ and $1_{K_\pi} \otimes \lambda$ are unitarily conjugate. Indeed, define $U : K_\pi \otimes \ell^2(\Gamma) \to K_\pi \otimes \ell^2(\Gamma)$ by

$$U(\xi \otimes \delta_t) = \pi_t \xi \otimes \delta_t.$$

It is routine to check that U is a unitary and $U(1_{K_\pi} \otimes \lambda_s)U^* = \pi_s \otimes \lambda_s$, for every $s \in \Gamma$. Therefore, the left multiplication extends to M. Denote by $1_\Gamma : \Gamma \to \mathcal{U}(\mathbf{C})$ the trivial representation.

Proposition 11 (Representations and Bimodules)**.** *The formulae above endow the Hilbert space $\mathcal{H}(\pi) = K_\pi \otimes \ell^2(\Gamma)$ with a structure of M-M-bimodule. Moreover, we have the following:*

(1) ${}_M\mathcal{H}(1_\Gamma)_M \cong {}_M\mathrm{L}^2(M)_M$ *and* ${}_M\mathcal{H}(\lambda_\Gamma)_M \cong {}_M(\mathrm{L}^2(M) \otimes \mathrm{L}^2(M))_M$.
(2) *For all unitary Γ-representations π_1 and π_2 such that $\pi_1 \subset_{\rm weak} \pi_2$, we have*
$${}_M\mathcal{H}(\pi_1)_M \subset_{\rm weak} {}_M\mathcal{H}(\pi_2)_M.$$
(3) *Whenever π_1 and π_2 are unitary Γ-representations, we have*
$${}_M\mathcal{H}(\pi_1 \otimes \pi_2)_M \cong {}_M(\mathcal{H}(\pi_1) \otimes_M \mathcal{H}(\pi_2))_M.$$

Proof. The proof is left as an exercise. ◻

Jones's basic construction. Throughout this section, we will denote by M a tracial von Neumann algebra with a distinguished faithful normal trace τ. Let $B \subset M$ be a unital von Neumann subalgebra. We always endow B with the restricted trace, that is, $\tau_B = \tau|B$.

Proposition 12. *There exists a unique trace-preserving conditional expectation $E_B : M \to B$.*

Proof. Denote by $e_B : \mathrm{L}^2(M) \to \mathrm{L}^2(B)$ the orthogonal projection. Let $x \in M$. Then we have

$$\|e_B(\widehat{x})b\| = \|e_B(\widehat{x}b)\| = \|e_B(\widehat{xb})\| \leq \|\widehat{xb}\| = \|xb\|_2 \leq \|x\|_\infty \|b\|_2.$$

It follows that $e_B(\widehat{x}) \in \mathbf{B}(\mathrm{L}^2(B))$. Since we moreover have $e_B(\widehat{x}) \in JBJ'$, we get $e_B(\widehat{x}) \in B$. The mapping $E_B : M \to B : x \mapsto e_B(\widehat{x})$ is the conditional expectation. The rest of the proof is left to the reader. □

The *basic construction* $\langle M, e_B\rangle$ is the von Neumann subalgebra of $\mathbf{B}(\mathrm{L}^2(M))$ generated by M and the projection e_B. Observe that $Je_B = e_B J$ and $e_B x e_B = E_B(x)e_B$ for all $x \in M$.

Proposition 13. *The following are true.*

1. $\langle M, e_B\rangle = (JBJ)' \cap \mathbf{B}(\mathrm{L}^2(M))$.
2. *The central support of e_B in $\langle M, e_B\rangle$ equals 1. In particular, the $*$-subalgebra generated by Me_BM is strongly dense in $\langle M, e_B\rangle$.*
3. *The conditional expectation $E_B : M \to B$ extends to $\langle M, e_B\rangle$ by the formula $e_B x e_B = E_B(x)e_B$.*
4. *$\langle M, e_B\rangle$ is endowed with a semifinite faithful normal trace defined by*

$$\mathrm{Tr}(xe_By) = \tau(xy), \forall x, y \in M.$$

Proof. (1) For $x \in B$, we clearly have $x\mathrm{L}^2(B) \subset \mathrm{L}^2(B)$ and $x\mathrm{L}^2(B)^\perp \subset \mathrm{L}^2(B)^\perp$, hence $xe_B = e_Bx$. If $x \in M \cap \{e_B\}'$, then

$$E_B(x)\widehat{1} = e_B(x\widehat{1}) = e_Bx\widehat{1} = xe_B(\widehat{1}) = x\widehat{1}.$$

Therefore $x = E_B(x) \in B$. It follows that $B = M \cap \{e_B\}'$. Thus,

$$(JBJ)' = JB'J = \langle JM'J, Je_BJ\rangle = \langle M, e_B\rangle.$$

(2) The map $B \ni x \mapsto xe_B \in Be_B$ is a $*$-isomorphism. Indeed, if $xe_B = 0$, then $x\eta = 0$, for every $\eta \in \mathrm{L}^2(B)$. Since $x \in B$, it follows that $x = 0$. Denote by $z(e_B)$ the central support of e_B in B'. Then $z(e_B) \in B$ and $z(e_B)e_B = e_B$. Hence $z(e_B) = 1$. Thus the central support of $e_B = Je_BJ$ in $JB'J$ is equal to 1. It is clear that $\mathcal{I} = \mathrm{span}(Me_BM)$ is a $*$-subalgebra of $\langle M, e_B\rangle$ and a two-sided ideal of the $*$-algebra generated by M and e_B. Thus $\overline{\mathcal{I}}$ is a strongly closed two-sided ideal of $\langle M, e_B\rangle$. Moreover

$$\mathcal{I}\,\mathrm{L}^2(M) = Me_B\,\mathrm{L}^2(M) = M\,\mathrm{L}^2(B) \supset M\,\widehat{1}.$$

Since $\mathcal{I}$ acts non-degenerately, we get $\overline{\mathcal{I}} = \langle M, e_B\rangle$.

(3) We have that $e_B(\text{span}(Me_BM))e_B \subset Be_B$. Since $\text{span}(Me_BM)$ is strongly dense in $\langle M, e_B\rangle$, it follows that $e_B\langle M, e_B\rangle e_B = Be_B$. For all $T \in \langle M, e_B\rangle$, denote by $\Phi_B(T)$ the unique element of B such that $e_BTe_B = \Phi_B(T)e_B$. Then $\Phi_B : \langle M, e_B\rangle \to B$ is a conditional expectation which extends $E_B : M \to B$.

(4) Since e_B has central support 1 in $\langle M, e_B\rangle$, one can find partial isometries $v_i\langle M, e_B\rangle$ such that $v_i^*v_i \leq e_B$ and $\sum_i v_iv_i^* = 1$. It follows that

$$\bigoplus_i v_i\mathrm{L}^2(B) = \mathrm{L}^2(M).$$

Define the following normal weight Tr on $\langle M, e_B\rangle$ by

$$\mathrm{Tr}(x) = \sum_i \langle xv_i\widehat{1}, v_i\widehat{1}\rangle, \forall x \in \langle M, e_B\rangle_+.$$

Assume that $\mathrm{Tr}(x^*x) = 0$. Then $xv_i\widehat{1} = 0$, for every i. For every $b \in B$, we have

$$xv_ib\widehat{1} = xv_iJb^*J\widehat{1} = Jb^*Jxv_i\widehat{1} = 0.$$

Therefore $x = 0$ and Tr is faithful. For every $x, y \in M$, we have

$$\begin{aligned}
\mathrm{Tr}(xe_By) &= \sum_i \langle xe_Byv_i\widehat{1}, v_i\widehat{1}\rangle = \sum_i \langle e_Byv_ie_B\widehat{1}, e_Bx^*v_ie_B\widehat{1}\rangle \\
&= \sum_i \langle E_B(yv_i)e_B\widehat{1}, E_B(x^*v_i)e_B\widehat{1}\rangle = \sum_i \tau(E_B(x^*v_i)^*E_B(yv_i)) \\
&= \sum_i \tau(E_B(v_i^*y^*)^*E_B(v_i^*x)) = \sum_i \langle E_B(v_i^*x)e_B\widehat{1}, E_B(v_i^*y^*)e_B\widehat{1}\rangle \\
&= \sum_i \langle e_Bv_i^*xe_B\widehat{1}, e_Bv_i^*y^*e_B\widehat{1}\rangle = \sum_i \langle v_iv_i^*x\widehat{1}, y^*\widehat{1}\rangle \\
&= \langle \sum_i v_iv_i^*x\widehat{1}, y^*\widehat{1}\rangle = \langle x\widehat{1}, y^*\widehat{1}\rangle = \tau(yx) = \tau(xy).
\end{aligned}$$

We get that Tr is semifinite since $\text{span}(Me_BM)$ is a strongly dense $*$-subalgebra in $\langle M, e_B\rangle$. For every $x, y, z, t \in \langle M, e_B\rangle$, we have

$$\begin{aligned}
\mathrm{Tr}(xe_By\, ze_Bt) &= \mathrm{Tr}(xE_B(yz)e_Bt) = \tau(xE_B(yz)t) \\
&= \tau(E_B(yz)E_B(tx)) = \tau(zE_B(tx)y) \\
&= \mathrm{Tr}(zE_B(tx)e_By) = \mathrm{Tr}(ze_Bt\, xe_By).
\end{aligned}$$

Thus Tr is a trace. This completes the proof. □

It follows from the previous proposition that

$$\langle M, e_B\rangle = \{T \in \mathbf{B}(\mathrm{L}^2(M)) : T(\xi b) = T(\xi)b, \forall \xi \in \mathrm{L}^2(M), \forall b \in B\}.$$

Let $\mathcal{H}_B$ be a right B-submodule of $\mathrm{L}^2(M)_B$. Write $P_{\mathcal{H}} : \mathrm{L}^2(M) \to \mathcal{H}$ for the orthogonal projection. It is clear that $P_{\mathcal{H}} \in \langle M, e_B\rangle$. We define the *von Neumann dimension* of $\mathcal{H}_B$ by $\dim(\mathcal{H}_B) = \mathrm{Tr}(P_{\mathcal{H}})$.

We will need the following useful fact.

Proposition 14. *Let (N, Tr) be a semifinite von Neumann algebra. Let $\mathcal{C} \subset N$ be a σ-weakly closed subset which is bounded for both the uniform norm $\|\cdot\|_\infty$ and the L^2-norm $\|\cdot\|_{2,\mathrm{Tr}}$. Write $\widehat{\cdot} : \mathcal{C} \to \mathrm{L}^2(N, \mathrm{Tr})$ for the canonical inclusion. Then $\widehat{\mathcal{C}}$ is a weakly closed subset of $\mathrm{L}^2(N, \mathrm{Tr})$.*

Proof. Let $\xi \in \mathrm{L}^2(N, \mathrm{Tr})$ and $x_n \in \mathcal{C}$ a net such that $\lim_n \langle \widehat{x}_n - \xi, \eta \rangle = 0$ for all $\eta \in \mathrm{L}^2(N, \mathrm{Tr})$. Since $\mathcal{C}$ is uniformly bounded and σ-weakly closed, passing to a subnet, we may assume that there exists $x \in \mathcal{C}$ such that $x_n \to x$ for the σ-WOT.

Since (N, Tr) is semifinite, choose an increasing sequence of projections p_k such that $\lim_k p_k = 1$ for the SOT and $\mathrm{Tr}(p_k) < \infty$. Observe that the image of $\bigcup_k p_k N p_k$ under $\widehat{\cdot}$ is L^2-dense in $\mathrm{L}^2(N, \mathrm{Tr})$. Moreover, for every $k \in \mathbf{N}$ and every $y \in N$, we have

$$\begin{aligned}
\lim_n \langle \widehat{x} - \widehat{x}_n, \widehat{p_k y p_k} \rangle_{\mathrm{L}^2} &= \lim_n \mathrm{Tr}(p_k y^* p_k (x - x_n)) \\
&= \lim_n \mathrm{Tr}(y^* p_k (x - x_n) p_k) \\
&= \lim_n \langle (x - x_n) \widehat{p}_k, \widehat{p_k y} \rangle_{\mathrm{L}^2} = 0.
\end{aligned}$$

This implies that $\widehat{x}_n \to \widehat{x}$ weakly in $\mathrm{L}^2(N, \mathrm{Tr})$ and so $\xi = \widehat{x}$. This shows that $\widehat{\mathcal{C}}$ is weakly closed in $\mathrm{L}^2(N, \mathrm{Tr})$. ◻

Popa's intertwining techniques. The aim of this section is to prove the following powerful method to intertwine subalgebras in a given tracial von Neumann algebra (M, τ).

Theorem 5 (Popa). *Let (M, τ) be a tracial von Neumann algebra. Let $A \subset 1_A M 1_A$ and $B \subset 1_B M 1_B$ be von Neumann subalgebras. The following are equivalent:*

(1) *There exist projections $p \in A$, $q \in B$, a nonzero partial isometry $v \in pMq$ and a unital normal $*$-homomorphism $\theta : pAp \to qBq$ such that $xv = \theta(x)v$ for every $x \in pAp$.*
(2) *There is no net of unitaries $w_n \in \mathcal{U}(A)$ such that $\lim_n \|E_B(x^* w_n y)\|_2 = 0$ for all $x, y \in 1_A M 1_B$.*

Proof. To simplify the notation, we will assume that $A, B \subset M$ are *unital* von Neumann subalgebras, that is, $1_A = 1_B = 1$.

(1) $\Rightarrow$ (2) By approximating the central support of $p \in A$, we may choose partial isometries $u_1, \dots, u_k \in A$ such that $u_i^* u_i \leq p$ and $\sum_{i=1}^k u_i u_i^* = z \in \mathcal{Z}(A)$. Define the normal $*$-homomorphism $\Theta : Az \to \mathbf{M}_k(qBq)$ by $\Theta(x) = [\theta(u_i^* x u_j)]_{ij}$ and the partial isometry $V = [u_1 v \cdots u_k v] \in \mathbf{M}_{1,k}(\mathbf{C}) \otimes pMq$. We have $xV = V\Theta(x)$ for all $x \in Az$.

Observe that $\Theta(z) = \mathrm{Diag}(\theta(u_i^* u_i))$ and $V^* V = \mathrm{Diag}(v^* u_i^* u_i v) \leq \Theta(z)$. Then for every $w \in \mathcal{U}(A)$, we get

$$\begin{aligned}\|E_{\Theta(Az)}(V^*V)\|_2 &= \|E_{\Theta(Az)}(V^*V)\Theta(wz)\|_2 \\ &= \|E_{\Theta(Az)}(V^*V\Theta(wz))\|_2 \\ &= \|E_{\Theta(Az)}(V^*wV)\|_2.\end{aligned}$$

If there would exist a net $w_n \in \mathcal{U}(A)$ such that $\lim_n \|E_B(x^* w_n y)\|_2 = 0$ for all $x, y \in M$, then we would have $\lim_n \|E_{\Theta(Az)}(V^* w_n V)\|_2 = 0$ and so $E_{\Theta(Az)}(V^*V) = 0$. This is impossible since $V \neq 0$.

(2) $\Rightarrow$ (1) There exist $\varepsilon > 0$ and a finite subset $\mathcal{F} \subset M$ such that

$$\sum_{x,y \in \mathcal{F}} \|E_B(x^* w y)\|_2^2 \geq \varepsilon^2$$

for all $w \in \mathcal{U}(A)$. Put $d = \sum_{x \in \mathcal{F}} x e_B x^* \in \langle M, e_B \rangle_+$. We have $\mathrm{Tr}(d) = \sum_{x \in \mathcal{F}} \tau(xx^*) < \infty$. Moreover, for all $w \in \mathcal{U}(A)$, we have

$$\sum_{y \in \mathcal{F}} \langle w^* d w \, \widehat{y}, \widehat{y} \rangle_{\mathrm{L}^2(M)} = \sum_{x,y \in \mathcal{F}} \langle w^* x e_B x^* w \, \widehat{y}, \widehat{y} \rangle_{\mathrm{L}^2(M)} = \sum_{x,y \in \mathcal{F}} \|E_B(x^* w y)\|_2^2 \geq \varepsilon^2.$$

Denote by $\mathcal{C}$ the σ-weak closure of the convex hull of $\{w^* d w : w \in \mathcal{U}(A)\}$. We get that $0 \notin \mathcal{C}$. Since $\mathcal{C}$ can be regarded as a closed bounded convex subset of $\mathrm{L}^2(\langle M, e_B \rangle, \mathrm{Tr})$ (see Proposition 14), denote by $c \in \mathcal{C}$ the unique circumcenter of $\mathcal{C}$. Since $w^* \mathcal{C} w = \mathcal{C}$, we get $w^* c w = c$ for all $w \in \mathcal{U}(A)$. Thus, we get $c \in A' \cap \langle M, e_B \rangle_+$ with $0 < \mathrm{Tr}(c) < \infty$.

Define the nonzero spectral projection $e = \mathbf{1}_{[\|c\|/2, \|c\|]}(c) \in A' \cap \langle M, e_B \rangle_+$. Since $\frac{\|c\|}{2} e \leq c\, e$, we have $\mathrm{Tr}(e) < \infty$. Let $\mathcal{H} = e\mathrm{L}^2(M)$. Then ${}_A\mathcal{H}_B$ is a nonzero A-B-subbimodule of ${}_A\mathrm{L}^2(M)_B$ such that $\dim(\mathcal{H}_B) = \mathrm{Tr}(e) < \infty$. Then there exist a nonzero projection $p \in A$ and a nonzero pAp-B-subbimodule $\mathcal{K} \subset p\mathcal{H}$ such that $\mathcal{K}$ is isomorphic as a right B-module to a right B-submodule of $\mathrm{L}^2(B)_B$.

Denote by $V : \mathcal{K}_B \to \mathrm{L}^2(B)_B$ the corresponding right B-bimodular isometry. Let $x \in pAp$. Since VxV^* commutes with the right B-action on $\mathrm{L}^2(B)$, we have $VxV^* \in qBq$ where $q = VV^*$. Therefore $\theta : pAp \to qBq$ defined by $\theta(x) = VxV^*$ is a unital normal $*$-homomorphism. Put $\xi = V^*\widehat{1} \in \mathcal{K}$. We have $\xi \neq 0$ since $V\xi = VV^*\widehat{1} = \widehat{q} \neq 0$. Moreover, for all $x \in pAp$, we have

$$x\xi = xV^*\widehat{1} = V^*\theta(x)\widehat{1} = V^*\widehat{1}\theta(x) = \xi\theta(x).$$

Since $\mathcal{K} \subset \mathrm{L}^2(M)$, we may regard ξ as a square summable closed operator affiliated with M. Write $\xi = v|\xi|$ for the polar decomposition of ξ. We have that $v \in pMq$, $v \neq 0$ and $|\xi|$ is a positive square summable closed operator affiliated with M. For all $u \in \mathcal{U}(pAp)$, we have

$$|\xi|^2 = (u\xi)^*(u\xi) = (\xi\theta(u))^*(\xi\theta(u)) = \theta(u)^*|\xi|^2\theta(u).$$

It follows that $|\xi|$ is affiliated with $\theta(pAp)' \cap qMq$. Moreover, for all $u \in \mathcal{U}(pAp)$, we have

$$uv|\xi| = u\xi = \xi\theta(u) = v|\xi|\theta(u) = v\theta(u)\theta(u)^*|\xi|\theta(u) = v\theta(u)|\xi|.$$

It follows that $xv = v\theta(x)$ for all $x \in pAp$. □

If one of the equivalent conditions of Theorem 5 is satisfied, we say that A *embeds into* B *inside* M and denote $A \preceq_M B$.

Exercise 3. Let $\Gamma \curvearrowright (X, \mu)$ be a pmp action. Put $B = \mathrm{L}^\infty(X)$ and $M = B \rtimes \Gamma$. Let $A \subset M$ be a von Neumann subalgebra. Show that the following are equivalent:

(1) $A \npreceq_M B$
(2) There exists a net $w_n \in \mathcal{U}(A)$ such that $\lim_n \|E_B(w_n u_s^*)\|_2 = 0$ for all $s \in \Gamma$.

In the case when $A, B \subset M$ are Cartan subalgebras, we can upgrade the previous result in order to obtain a genuine conjugation by a unitary. We first prove a technical result.

Lemma 3. *Let $A \subset M$ be a maximal abelian subalgebra of a tracial von Neumann algebra. For every projection $q \in M$, there exists a partial isometry $u \in M$ such that $u^*u \in A$ and $uu^* = q$.*

Proof. Let $q \in M$ be a nonzero projection. We start by proving that there exists a nonzero projection $p \in A$ and a partial isometry $u \in M$ such that $u^*u = p$ and $uu^* \leq q$. Observe that $\mathcal{Z}(M) \subset A \subset M$. Denote by $\mathrm{ctr} : M \to \mathcal{Z}(M)$ the center valued trace.

Since $q \neq 0$ and up to cutting down by a nonzero spectral projection of the form $\mathbf{1}_{[\varepsilon,1]}(\mathrm{ctr}(q)) \in \mathcal{Z}(M)$, we may assume that there exists $\varepsilon > 0$ such that $\mathrm{ctr}(q) \geq \varepsilon$. There are two cases to consider.

- Assume that $\mathcal{Z}(M)z = Az$ for some nonzero projection $z \in A$. Then we have $Az = zMz$ and so z is an abelian projection. Since q has central support equal to 1, there exists $u \in M$ such that $u^*u = z$ and $uu^* = q$.
- Assume that $\mathcal{Z}(M)z \neq Az$ for every nonzero projection $z \in A$. Then by Rohlin's classification of pmp factor maps, there exists a trace preserving $*$-isomorphism $\theta : A \to \mathcal{Z}(M) \mathbin{\overline{\otimes}} \mathrm{L}(\mathbf{Z})$ such that $\theta(z) = z \otimes 1$ for all $z \in \mathcal{Z}(M)$. Choose a projection $s \in \mathrm{L}(\mathbf{Z})$ of trace ε and put $p = \theta^{-1}(1 \otimes s) \in A$. Then $\mathrm{ctr}(p) = \varepsilon \leq \mathrm{ctr}(q)$. This implies that there exists a partial isometry $u \in M$ such that $u^*u = p$ and $uu^* \leq q$.

By Zorn's Lemma, choose a maximal family (p_i, q_i) with respect to inclusion of pairwise orthogonal projections $p_i \in A$ and pairwise orthogonal projections $q_i \leq q$ such that for all i there exists a partial isometry $u_i \in M$ which satisfies $p_i = u_i^* u_i$ and $q_i = u_i u_i^*$. The previous paragraph together with the maximality assumption show that $q = \sum_i q_i$. Letting $p = \sum_i p_i \in A$ and $u = \sum_i u_i \in M$, we get $p = u^* u$ and $q = uu^*$. □

Theorem 6 (Popa)**.** *Let $A, B \subset M$ be Cartan subalgebras in a II_1 factor. The following conditions are equivalent:*

(1) $A \preceq_M B$.
(2) *There exists $u \in \mathcal{U}(M)$ such that $uAu^* = B$.*

Proof. We only need to show that (1) ⇒ (2). Let $p \in A$, $q \in B$ be projections, $v \in pMq$ a non zero partial isometry and $\theta : Ap \to Bq$ a unital normal $*$-homomorphism such that $xv = v\theta(x)$ for all $x \in Ap$. Observe that $vv^* \in (Ap)' \cap pMp = Ap$ and $v^*v \in \theta(Ap)' \cap qMq$. Since $Bq \subset \theta(Ap)' \cap qMq$ is maximal abelian, there exists $u \in \theta(Ap)' \cap qMq$ such that $uu^* = v^*v$ and $u^*u \in Bq$ by Lemma 3. Put $w = vu$. We then have $ww^* = vuu^*v^* = vv^*$, $w^*w = u^*v^*vu = u^*u \in Bq$ and $xw = w\theta(x)$ for all $x \in Ap$. Therefore, we may assume that $v^*v = q$ and $vv^* = p$. We have $v^*Av \subset Bq$. Since A is maximal abelian, we have $Bq \subset v^*Av$ and thus $v^*Av = Bq$.

Next, we may shrink p so that $\tau(p) = 1/n$. Since M is a II_1 factor and $A, B \subset M$ are both Cartan subalgebras, by Theorem 4 we may choose partial isometries $u_i, v_i \in M$ such that for all $1 \leq i \leq n$, we have $p = u_i^* u_i$, $u_i u_i^* \in A$, $u_i^* A u_i = A u_i^* u_i$, $\sum_{i=1}^n u_i u_i^* = 1$, $u_1 = p$ and $q = v_i^* v_i$, $v_i v_i^* \in B$, $v_i^* B v_i = B v_i^* v_i$, $\sum_{i=1}^n v_i v_i^* = 1$, $v_1 = q$. Define $u = \sum_{i=1}^n v_i v^* u_i^* \in \mathcal{U}(M)$. We obtain $uAu^* = B$. □

Lecture 4

In the final lecture, we prove Connes's characterization of amenable tracial von Neumann algebras.

Preliminaries. For an inclusion of von Neumann algebra $M \subset \mathcal{N}$, we say that a state $\varphi \in \mathcal{N}^*$ is M-central if $\varphi(xT) = \varphi(Tx)$ for all $x \in M$ and all $T \in \mathcal{N}$. We will be using the following notation: for all $x \in M$, put $\overline{x} = (x^{\mathrm{op}})^* \in M^{\mathrm{op}}$.

Regarding $M \otimes_{\mathrm{alg}} M^{\mathrm{op}} \subset \mathbf{B}(\mathrm{L}^2(M) \otimes \mathrm{L}^2(M))$, we will denote by $\|\cdot\|_{\min}$ the operator norm on $M \otimes_{\mathrm{alg}} M^{\mathrm{op}}$ induced by $\mathbf{B}(\mathrm{L}^2(M) \otimes \mathrm{L}^2(M))$. It is called the *minimal tensor norm.*

Let H be a separable Hilbert space. For every $p \geq 1$, define the pth-*Schatten class* $\mathcal{S}_p(H)$ by

$$\mathcal{S}_p(H) = \{T \in \mathbf{B}(H) : \mathrm{Tr}(|T|^p) < \infty\}.$$

It is a Banach space with norm given by $\|T\|_p = \mathrm{Tr}(|T|^p)^{1/p}$. Observe that $\mathcal{S}_1(H)$ is the space of *trace-class* operators and $\mathcal{S}_2(H)$ is the (Hilbert) space of Hilbert-Schmidt operators. It is also denoted by $\mathrm{HS}(H)$.

Let M be a finite von Neumann algebra with a distinguished faithful normal trace τ. Observe that the unitary $U : \mathrm{HS}(\mathrm{L}^2(M)) \to \mathrm{L}^2(M)\otimes\mathrm{L}^2(M)$ defined by $U(\langle\cdot,\eta\rangle\xi) = \xi \otimes J\eta$ is an M-M-bimodule isomorphism.

We will be using the following technical results.

Lemma 4. *Let $\mathcal{A}$ be a unital C*-algebra, $u \in \mathcal{U}(\mathcal{A})$ and $\omega \in \mathcal{A}^*$ a state. Then we have*

$$\max\{\|\omega - \omega(u\cdot)\|, \|\omega - \omega(\cdot u^*)\|, \|\omega - \omega \circ \mathrm{Ad}(u)\|\} \leq 2\sqrt{2|1-\omega(u)|}.$$

Proof. Let $(\pi_\omega, \mathcal{H}_\omega, \xi_\omega)$ the GNS representation associated with the state ω on $\mathcal{A}$. Then $\omega(a) = \langle \pi_\omega(a)\xi_\omega, \xi_\omega\rangle$ for all $a \in \mathcal{A}$. We have

$$\|\omega - \omega(\cdot u^*)\| \leq \|\xi_\omega - \pi_\omega(u)^*\xi_\omega\| \leq \sqrt{2(1-\Re\omega(u))} \leq \sqrt{2|1-\omega(u)|}.$$

Likewise, we get $\|\omega - \omega(u\cdot)\| \leq \sqrt{2|1-\omega(u)|}$. Moreover, we have

$$\|\omega - \omega \circ \mathrm{Ad}(u)\| \leq 2\|\xi_\omega - \pi_\omega(u)^*\xi_\omega\| \leq 2\sqrt{2|1-\omega(u)|}. \qquad \square$$

The previous lemma implies in particular that when $\omega(u) = 1$, then

$$\omega = \omega(\cdot u^*) = \omega(u\cdot) = \omega \circ \mathrm{Ad}(u).$$

Lemma 5 (Powers-Størmer Inequality)**.** *Let H be a Hilbert space and $S, T \in \mathcal{S}_2(H)_+$. Then we have*

$$\|S-T\|_2^2 \leq \|S^2 - T^2\|_1 \leq \|S-T\|_2\|S+T\|_2.$$

Before starting the proof, we make the following observations:

- Whenever $A, B \in \mathbf{B}(H)$ have finite rank and if we write $AB = U|AB|$ for the polar decomposition, by the Cauchy-Schwarz Inequality, we have
$$\|AB\|_1 = \mathrm{Tr}(|AB|) = \mathrm{Tr}(U^*AB) \leq \|U^*A\|_2\|B\|_2 \leq \|A\|_2\|B\|_2.$$
- Whenever $A, B \in \mathbf{B}(H)_+$ and A or B has finite rank, we have $\mathrm{Tr}(AB) \geq 0$. Indeed, without loss of generality, we may assume that B has finite rank and we write $B = \sum_{i=1}^n \lambda_i\langle\cdot,\xi_i\rangle\,\xi_i$. Then $AB = \sum_{i=1}^n \lambda_i\langle\cdot,\xi_i\rangle\,A\xi_i$ and so $\mathrm{Tr}(AB) = \sum_{i=1}^n \lambda_i\langle A\xi_i,\xi_i\rangle \geq 0$.

Proof. First observe that using the Spectral Theorem, we may assume that S, T have both finite rank and still satisfy $S, T \geq 0$.

The identity

$$S^2 - T^2 = \frac{1}{2}((S+T)(S-T) + (S-T)(S+T))$$

together with the first observation give the right inequality.

Put $p = \mathbf{1}_{[0,+\infty)}(S-T)$. We have $(S-T)p \geq 0$ and $(T-S)p^\perp \geq 0$. Then using the previous identity together with the second observation twice, we have

$$\begin{aligned}
\|S-T\|_2^2 &= \mathrm{Tr}((S-T)^2) \\
&= \mathrm{Tr}((S-T)(S-T)p + (T-S)(T-S)p^\perp) \\
&\leq \mathrm{Tr}((S+T)(S-T)p + (T+S)(T-S)p^\perp) \\
&= \mathrm{Tr}((S^2-T^2)p + (T^2-S^2)p^\perp) \\
&\leq \mathrm{Tr}(|S^2-T^2|p + |T^2-S^2|p^\perp) \\
&= \mathrm{Tr}(|S^2-T^2|) = \|S^2-T^2\|_1. \qquad \square
\end{aligned}$$

Connes's theorem. This section is devoted to proving Connes's characterization of *amenability* for tracial von Neumann algebras.

Definition 10. Let (M,τ) be a tracial von Neumann algebra with separable predual. We say that M is *amenable* if there exists an M-central state $\varphi \in \mathbf{B}(\mathrm{L}^2(M))$ such that $\varphi|M = \tau$. We say that M is *hyperfinite* if there exists an increasing sequence of unital finite dimensional $*$-subalgebras $Q_n \subset M$ such that $M = \bigvee_n Q_n$.

Theorem 7 (Connes)**.** *Let (M,τ) be a tracial von Neumann algebra with separable predual. The following are equivalent:*

(1) *There exists a conditional expectation $E : \mathbf{B}(\mathrm{L}^2(M)) \to M$.*
(2) *There exists an M-central state φ on $\mathbf{B}(\mathrm{L}^2(M))$ such that $\varphi|M = \tau$.*
(3) *There exists a net of unit vectors $\xi_n \in \mathrm{L}^2(M) \otimes \mathrm{L}^2(M)$ such that $\lim_n \|x\xi_n - \xi_n x\|_2 = 0$ and $\lim_n \langle x\xi_n, \xi_n\rangle = \tau(x)$ for all $x \in M$.*
(4) ${}_M\mathrm{L}^2(M)_M \subset_{\text{weak}} {}_M(\mathrm{L}^2(M) \otimes \mathrm{L}^2(M))_M$.
(5) *For all $a_1, \dots, a_k, b_1, \dots, b_k \in M$, we have*

$$|\tau(\sum_{i=1}^k a_i b_i)| \leq \|\sum_{i=1}^k a_i \otimes b_i^{\mathrm{op}}\|_{\min}.$$

(6) *M is hyperfinite.*

Whenever $M = \mathrm{L}(\Gamma)$ is the von Neumann algebra of a countable discrete group, the previous conditions are equivalent to:

(7) *Γ is amenable.*

Proof. We show that (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) $\Leftrightarrow$ (7) and (6) $\Rightarrow$ (1). The proof of (1) $\Rightarrow$ (6) is beyond the scope of these notes.

(1) $\Rightarrow$ (2) Put $\varphi = \tau \circ E$.

(2) $\Rightarrow$ (3) and (1) Let φ be an M-central state on $\mathbf{B}(\mathrm{L}^2(M))$. Since the set of normal states is $\sigma(\mathbf{B}(\mathrm{L}^2(M))^*, \mathbf{B}(\mathrm{L}^2(M)))$-dense in the set of states, we may choose a net of normal states $(\varphi_j)_{j\in J}$ on $\mathbf{B}(\mathrm{L}^2(M))$ such that $\lim_J \varphi_j(T) =$

$\varphi(T)$ for all $T \in \mathbf{B}(\mathrm{L}^2(M))$. We get that $\varphi_j \circ \mathrm{Ad}(u) - \varphi_j \to 0$ with respect to the $\sigma(\mathbf{B}(\mathrm{L}^2(M))_*, \mathbf{B}(\mathrm{L}^2(M)))$-topology, for all $u \in \mathcal{U}(M)$. Using Hahn-Banach Theorem and up to replacing the net $(\varphi_j)_{j\in J}$ by a net $(\varphi'_k)_{k\in K}$ where each φ'_k is equal to a finite convex combination of some of the φ_j's, we may assume that $\|\varphi_j \circ \mathrm{Ad}(u) - \varphi_j\| \to 0$ for all $u \in \mathcal{U}(M)$. For every $j \in J$, let $T_j \in \mathcal{S}_1(\mathrm{L}^2(M))_+$ be the unique trace-class operator such that $\varphi_j(S) = \mathrm{Tr}(T_j S)$ for all $S \in \mathbf{B}(\mathrm{L}^2(M))$. We get $\|T_j\|_1 = 1$ and $\lim_J \|uT_j u^* - T_j\|_{1,\mathrm{Tr}} = 0$ for all $u \in \mathcal{U}(M)$. Put $\xi_j = T_j^{1/2} \in \mathcal{S}_2(\mathrm{L}^2(M))$ and observe that $\|\xi_j\|_2 = 1$. Since ξ_j is a Hilbert-Schmidt operator, we may regard $\xi_j \in \mathrm{L}^2(M) \otimes \mathrm{L}^2(M)$. By the Powers-Størmer Inequality, we get $\lim_J \|u\xi_j u^* - \xi_j\|_2 = 0$ for all $u \in \mathcal{U}(M)$. Moreover, we have

$$\lim_J \langle x\xi_j, \xi_j\rangle = \lim_J \mathrm{Tr}(T_j x) = \lim_J \varphi_j(x) = \varphi(x) = \tau(x), \forall x \in M.$$

This proves (3). In order to show (1), let $a \in M$ and $T \in \mathbf{B}(\mathrm{L}^2(M))$. Write $a = v|a|$ for the polar decomposition of a in M. Then we have

$$\begin{aligned}
|\varphi(aT)| &= |\varphi(|a|^{1/2} T v |a|^{1/2})| \\
&= |\lim_J \langle |a|^{1/2} T v |a|^{1/2} \xi_j, \xi_j\rangle| \\
&= |\lim_J \langle T v |a|^{1/2} \xi_j, |a|^{1/2} \xi_j\rangle| \\
&\leq \limsup_J \|T v |a|^{1/2} \xi_j\| \, \||a|^{1/2} \xi_j\| \\
&\leq \|T\|_\infty \limsup_J \||a|^{1/2} \xi_j\|^2 \\
&= \|T\|_\infty \|a\|_1.
\end{aligned}$$

Therefore the functional $a \mapsto \varphi(aT)$ is bounded for the L^1-norm and thus there exists $E(T) \in M$ such that $\varphi(aT) = \tau(aE(T))$ for all $a \in M$ and all $T \in \mathbf{B}(\mathrm{L}^2(M))$. Then $E : \mathbf{B}(\mathrm{L}^2(M)) \to M$ is a conditional expectation.

(3) $\Rightarrow$ (4) Let $a_1, \dots, a_k, b_1, \dots, b_k \in M$ and put $T = \sum_{i=1}^k a_i \otimes b_i^{\mathrm{op}}$. Let $c, d \in M$. Then

$$\begin{aligned}
|\langle \pi_{\mathrm{L}^2(M)}(T)\widehat{c}, \widehat{d^*}\rangle| &= |\tau(\sum_{i=1}^k d a_i c b_i)| = \lim_n |\langle \sum_{i=1}^k d a_i c b_i \, \xi_n, \xi_n\rangle| \\
&= \lim_n |\langle \sum_{i=1}^k a_i \, \xi_n c \, b_i, d^* \xi_n\rangle| \\
&\leq \|\pi_{\mathrm{L}^2(M)\otimes \mathrm{L}^2(M)}(T)\|_\infty \lim_n \|\xi_n c\| \lim_n \|d^* \xi_n\| \\
&= \|\pi_{\mathrm{L}^2(M)\otimes \mathrm{L}^2(M)}(T)\|_\infty \|c\|_2 \|d^*\|_2.
\end{aligned}$$

This implies that $\|\pi_{\mathrm{L}^2(M)}(T)\|_\infty \leq \|\pi_{\mathrm{L}^2(M)\otimes \mathrm{L}^2(M)}(T)\|_\infty$.

(4) ⇒ (5) Let $a_1, \dots, a_k, b_1, \dots, b_k \in M$ and put $T = \sum_{i=1}^k a_i \otimes b_i^{\mathrm{op}}$. Since $\mathrm{L}^2(M) \otimes \mathrm{L}^2(M)$ is a left $M \overline{\otimes} M^{\mathrm{op}}$-module, we have

$$\|\pi_{\mathrm{L}^2(M)\otimes \mathrm{L}^2(M)}(T)\|_\infty = \|\sum_{i=1}^k a_i \otimes b_i^{\mathrm{op}}\|_{\min}.$$

Since by assumption, we have $\|\pi_{\mathrm{L}^2(M)}(T)\|_\infty \leq \|\pi_{\mathrm{L}^2(M)\otimes \mathrm{L}^2(M)}(T)\|_\infty$, we get

$$|\tau(\sum_{i=1}^k a_i b_i)| = |\langle \pi_{\mathrm{L}^2(M)}(T)\widehat{1}, \widehat{1}\rangle| \leq \|\pi_{\mathrm{L}^2(M)}(T)\|_\infty \leq \|\sum_{i=1}^k a_i \otimes b_i^{\mathrm{op}}\|_{\min}.$$

(5) ⇒ (2) Denote by $\Omega : M \otimes_{\mathrm{alg}} M^{\mathrm{op}} \to \mathbf{C}$ the $\|\cdot\|_{\min}$-bounded functional such that $\Omega(a \otimes b^{\mathrm{op}}) = \tau(ab)$. By the Hahn-Banach Theorem and since $M \otimes_{\mathrm{alg}} M^{\mathrm{op}} \subset \mathbf{B}(\mathrm{L}^2(M) \otimes \mathrm{L}^2(M))$, we may extend the functional Ω to $\mathbf{B}(\mathrm{L}^2(M) \otimes \mathrm{L}^2(M))$ without increasing the norm of Ω. We still denote this extension by Ω. Since $\|\Omega\| = 1 = \Omega(1)$, Ω is a state on $\mathbf{B}(\mathrm{L}^2(M) \otimes \mathrm{L}^2(M))$. Since $\Omega(u \otimes \overline{u}) = \tau(uu^*) = 1$ for all $u \in \mathcal{U}(M)$, we have

$$\Omega(S(u \otimes \overline{u})) = \Omega(S) = \Omega((u \otimes \overline{u})S)$$

for all $S \in \mathbf{B}(\mathrm{L}^2(M) \otimes \mathrm{L}^2(M))$ and all $u \in \mathcal{U}(M)$ (see Lemma 4).

Put $\varphi(T) = \Omega(T \otimes 1^{\mathrm{op}})$ for all $T \in \mathbf{B}(\mathrm{L}^2(M))$. Observe that $\varphi(x) = \Omega(x \otimes 1^{\mathrm{op}}) = \tau(x)$ for all $x \in M$. Moreover, for all $T \in \mathbf{B}(\mathrm{L}^2(M))$ and all $u \in \mathcal{U}(M)$, we have

$$\begin{aligned}\varphi(uT) &= \Omega(uT \otimes 1^{\mathrm{op}}) = \Omega((u \otimes \overline{u})(T \otimes u^{\mathrm{op}})) \\ &= \Omega((T \otimes u^{\mathrm{op}})(u \otimes \overline{u})) = \Omega(Tu \otimes 1^{\mathrm{op}}) \\ &= \varphi(Tu).\end{aligned}$$

(6) ⇒ (1) Assume that $M = \bigvee_n Q_n$ with $Q_n \subset M$ an increasing sequence of unital finite dimensional $*$-subalgebras. Denote by μ_n the unique Haar probability measure on the compact group $\mathcal{U}(Q_n)$. Choose a free ultrafilter ω on $\mathbf{N}$. For all $T \in \mathbf{B}(\mathrm{L}^2(M))$, put

$$\Phi(T) = \lim_{n \to \omega} \int_{\mathcal{U}(Q_n)} uTu^* \, \mathrm{d}\mu_n(u).$$

Then $E : \mathbf{B}(\mathrm{L}^2(M)) \to M$ defined by $E(T) = J\Phi(T)J$ is a conditional expectation.

Put $M = \mathrm{L}(\Gamma)$ and denote by $u_s \in M$ the canonical unitaries.

(1) ⇒ (7) Let $\varphi \in \mathbf{B}(\ell^2(\Gamma))^*$ be an $\mathrm{L}(\Gamma)$-central state such that $\varphi|\mathrm{L}(\Gamma) = \tau$. Define a state $m \in \ell^\infty(\Gamma)^*$ by $m = \varphi|\ell^\infty(\Gamma)$. Then m is an invariant mean and Γ is amenable.

(7) ⇒ (1) Assume that there exists a sequence of unit vectors $\zeta_n \in \ell^2(\Gamma)$ such that $\|\lambda_s \zeta_n - \zeta_n\| = 0$ for all $s \in \Gamma$. Put $M = \mathrm{L}(\Gamma)$. Consider the M-M-bimodule $\mathcal{H}_\lambda$ as defined in Lecture 3. Recall that ${}_M\mathcal{H}_{\lambda M} \cong {}_M(\mathrm{L}^2(M) \otimes$

$\mathrm{L}^2(M))_M$. Put $\xi_n = \zeta_n \otimes \widehat{1}$ and regard $\xi_n \in \mathrm{HS}(\mathrm{L}^2(M))$. Observe that $\lim_n \|u_s\xi_n - \xi_n u_s\| = 0$ all $s \in \Gamma$ and $\langle x\xi_n, \xi_n\rangle = \tau(x)$ for all $n \in \mathbf{N}$ and all $x \in M$.

Choose a free ultrafilter ω on $\mathbf{N}$ and put $\varphi(T) = \lim_\omega \langle T\xi_n, \xi_n\rangle$ for all $T \in \mathbf{B}(\mathrm{L}^2(M))$. We have $\varphi(u_s T) = \varphi(T u_s)$ for all $T \in \mathbf{B}(\mathrm{L}^2(M))$ and all $s \in \Gamma$ and $\varphi|M = \tau$. Let $x \in M$ and write $x = \sum_{s\in\Gamma} x_s u_s$ for its Fourier expansion. Put $x_{\mathcal{F}} = \sum_{s\in\mathcal{F}} x_s u_s \in \mathbf{C}[\Gamma]$ for $\mathcal{F} \subset \Gamma$ finite subset. By Cauchy-Schwarz Inequality, we have

$$|\varphi((x-x_{\mathcal{F}})T)| \leq \varphi((x-x_{\mathcal{F}})(x-x_{\mathcal{F}})^*)^{1/2}\,\varphi(T^*T)^{1/2} = \|x-x_{\mathcal{F}}\|_2\,\varphi(T^*T)^{1/2}$$

and so $\lim_{\mathcal{F}} \varphi(x_{\mathcal{F}}T) = \varphi(xT)$. Likewise, we have $\lim_{\mathcal{F}} \varphi(Tx_{\mathcal{F}}) = \varphi(Tx)$. This implies that $\varphi(xT) = \varphi(Tx)$ for all $x \in M$ and all $T \in \mathbf{B}(\mathrm{L}^2(M))$. $\square$

Exercise 4. Let $\Gamma \curvearrowright (X,\mu)$ be a pmp action of a countable discrete group on a standard probability space. Show that $\mathrm{L}^\infty(X) \rtimes \Gamma$ is amenable if and only if Γ is amenable.

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