AN INVITATION TO VON NEUMANN ALGEBRAS
LECTURE NOTES

CYRIL HOUDAYER

Abstract. These are the lecture notes of a graduate course given at the Université Paris-Sud (Orsay) in the Winter of 2017. In Section 1, we review some preliminary background on C^*-algebras. In Section 2, we review weak and strong operator topologies on $\mathbf{B}(H)$ and prove the spectral theorem for bounded normal operators. In Section 3, we introduce von Neumann algebras and prove some basic properties. In Section 4, we present two important classes of von Neumann algebras, namely group von Neumann algebras and Murray–von Neumann's group measure space constructions. Finally, in Section 5, we prove Connes's characterization of amenable tracial von Neumann algebras.

Contents

1. Preliminary background on C^*-algebras

All the algebras we consider are always over the field $\mathbf{C}$ of complex numbers.

1.1. Introduction to C^*-algebras.

1.1.1. *Definition and first properties.*

Definition 1.1. A C^*-*algebra* A is a Banach algebra endowed with an involution $A \to A : a \mapsto a^*$ which satisfies the following relation:

$$\|a^* a\| = \|a\|^2, \forall a \in A.$$

If A admits a unit, we say that A is a *unital* C^*-algebra. Denote by $\mathbf{B}(H)$ the Banach algebra of all bounded linear operators $T : H \to H$ endowed with the *supremum norm*:

$$\|T\|_\infty = \sup_{\|\xi\| \leq 1} \|T\xi\|.$$

Let $T \in \mathbf{B}(H)$. The *adjoint operator* T^* is defined by

$$\langle T\xi, \eta \rangle = \langle \xi, T^*\eta \rangle, \forall \xi, \eta \in H.$$

Examples 1.2. Here are examples of C*-algebras.

(1) Norm closed $*$-subalgebras of $\mathbf{B}(H)$.
(2) The space of all complex-valued continuous functions $\mathrm{C}(X)$ over a compact topological space X endowed with the supremum norm given by $\|f\|_\infty = \sup_{x\in X}|f(x)|$. The involution is given by $f^*(x) = \overline{f(x)}$ for all $x \in X$.
(3) Let Γ be a countable discrete group and let $\lambda : \Gamma \to \mathcal{U}(\ell^2(\Gamma))$ be the *left regular representation* defined by $\lambda_g\delta_h = \delta_{gh}$ for all $g, h \in \Gamma$. The *reduced group* C*-*algebra* $\mathrm{C}^*_\lambda(\Gamma)$ is defined as the norm closure of the linear span of $\{\lambda_g : g \in \Gamma\}$.

From now on, to avoid any technical difficulties, we will always assume that all C*-algebras are unital. Moreover, all $*$-homomorphisms are assumed to be unital. For $a \in A$, the *spectrum* of a is defined as follows:

$$\sigma(a) := \{\lambda \in \mathbf{C} : a - \lambda 1 \text{ is not invertible}\}.$$

Proposition 1.3. *For all $a \in A$, $\sigma(a)$ is a nonempty compact subset of* $\mathbf{C}$.

Proof. It is clear that $\sigma(a)$ is closed. Moreover for all $|\lambda| > \|a\|$, $1 - \lambda^{-1}a$ is invertible with inverse $\sum_n \lambda^{-n}a^n$. It follows that $\sigma(a)$ is bounded by $\|a\|$, whence $\sigma(a)$ is compact.

By contradiction, assume that $\sigma(a)$ is the empty set. Then the function $\lambda \mapsto (a - \lambda 1)^{-1}$ is entire and vanishing at infinity. By Hahn–Banach and Liouville Theorems, we get that this function is zero everywhere. Thus $a^{-1} = 0$, which is a contradiction. Thus $\sigma(a)$ is nonempty and compact. ☐

Observe that the above proof works more generally for any unital Banach algebra. We have the following useful corollary.

Corollary 1.4. *Any unital Banach algebra A in which every nonzero element is invertible is isomorphic to* $\mathbf{C}$.

Proof. Let $x \in A$ and choose $\lambda \in \sigma(a)$. Since $x - \lambda 1$ is not invertible, we have $x - \lambda 1 = 0$. Thus $A = \mathbf{C}1$. ☐

Exercise 1.5. Show that $\sigma(ab) \cup \{0\} = \sigma(ba) \cup \{0\}$, for all $a, b \in A$.

Exercise 1.6. Let A be a unital abelian Banach algebra and $\mathfrak{m} \subset A$ a proper ideal, that is, $1 \notin \mathfrak{m}$. Show that

$$\inf\{\|1 - x\| : x \in \mathfrak{m}\} \geq 1.$$

Deduce that the closure of any proper ideal is still proper and any maximal proper ideal is closed.

The *spectral radius* is defined by

$$r(a) := \sup\{|\lambda| : \lambda \in \sigma(a)\}.$$

We have $r(a) \leq \|a\|$.

Proposition 1.7. *For all $a \in A$, the sequence $(\|a^n\|^{1/n})_n$ converges to $r(a)$.*

Proof. If $\lambda \in \sigma(a)$, then $\lambda^n \in \sigma(a^n)$. Thus $|\lambda| \le \|a^n\|^{1/n}$, for all $n \in \mathbf{N}$. It follows that $|\lambda| \le \liminf \|a^n\|^{1/n}$ and hence $r(a) \le \liminf_n \|a^n\|^{1/n}$. Next, for $|z| < r(a)^{-1}$, $f : z \mapsto (1 - za)^{-1}$ is a holomorphic function which coincides with the power series $\sum_n z^n a^n$ when moreover $|z| < \|a\|^{-1}$. Observe that this power series represents f on the open disk with center 0 and radius $r(a)^{-1}$. However, this series cannot converge for $|z| > (\limsup \|a^n\|^{1/n})^{-1}$. Thus, we get that $\limsup \|a^n\|^{1/n} \le r(a)$. □

In particular, if $a, b \in A$ are commuting elements, we have that

$$\begin{aligned} r(ab) &= \lim \|(ab)^n\|^{1/n} = \lim \|a^n b^n\|^{1/n} \\ &\le \lim \|a^n\|^{1/n} \lim \|b^n\|^{1/n} \\ &= r(a)r(b). \end{aligned}$$

We say that a is *selfadjoint* if $a^* = a$; *normal* if $a^*a = aa^*$; *unitary* if $a^*a = aa^* = 1$. The group of unitaries is denoted by $\mathcal{U}(A)$. The subspace of selfadjoint elements in A is sometimes denoted by $\Re(A)$. For any subset $\mathcal{V} \subset A$, the unit ball of $\mathcal{V}$ will be denoted by $(\mathcal{V})_1$.

Proposition 1.8. *Let $a \in A$. The following are true.*

1. *If a is invertible, a^* is invertible and $(a^*)^{-1} = (a^{-1})^*$*
2. *a can be uniquely decomposed $a = x + \mathrm{i}y$, with x, y selfadjoint elements.*
3. *If a is a unitary then $\|a\| = 1$.*
4. *If a is normal then $\|a\| = r(a)$.*
5. *If B is another C^*-algebra and $\varphi : A \to B$ is a $*$-homomorphism then $\|\varphi(a)\| \le \|a\|$.*

Proof. We leave $(1), (2), (3)$ as an exercise. To prove (4), first assume that a is selfadjoint. One has $\|a^{2^n}\| = \|a\|^{2^n}$ for all $n \in \mathbf{N}$. Thus, $r(a) = \lim_n \|a^{2^n}\|^{2^{-n}} = \|a\|$. If a is normal, $\|a\|^2 = \|a^*a\| = r(a^*a) \le r(a^*)r(a) \le \|a^*\|\|a\| = \|a\|^2$, whence $r(a) = \|a\|$. To prove (5), let $a \in A$. Then

$$\|\varphi(a)\|^2 = \|\varphi(a)^*\varphi(a)\| = \|\varphi(a^*a)\| = r(\varphi(a^*a)) \le r(a^*a) = \|a^*a\| = \|a\|^2. \qquad \square$$

Corollary 1.9. *Any onto $*$-isomorphism $\varphi : A \to B$ is isometric.*

1.1.2. *Continuous functional calculus.*

Lemma 1.10. *Let $\chi : A \to \mathbf{C}$ be a unital algebraic homomorphism. Then the following assertions hold true.*

1. *For all $a \in A$, $|\chi(a)| \le \|a\|$.*
2. *For all $a \in \Re(A)$, $\chi(a) \in \mathbf{R}$.*
3. *For all $a \in A$, $\chi(a^*) = \overline{\chi(a)}$.*
4. *For all $a \in A$, $\chi(a^*a) \ge 0$.*
5. *For all $a \in \mathcal{U}(A)$, $|\chi(a)| = 1$.*

Proof. (1) For all $a \in A$, $\chi(a - \chi(a)1) = 0$, whence $a - \chi(a)1$ is not invertible. We get $\chi(a) \in \sigma(a)$ and so $|\chi(a)| \le \|a\|$.

(2) Assume that $a \in A$ is selfadjoint. Let $t \in \mathbf{R}$.

$$|\chi(a + \mathrm{i}t)|^2 \le \|a + \mathrm{i}t\|^2 = \|(a + \mathrm{i}t)^*(a + \mathrm{i}t)\| = \|(a - \mathrm{i}t)(a + \mathrm{i}t)\| \le \|a\|^2 + t^2.$$

Write $\chi(a) = \alpha + \mathrm{i}\beta$. We then get

$$\|a\|^2 + t^2 \ge |\alpha + \mathrm{i}(\beta + t)|^2 = \alpha^2 + \beta^2 + 2\beta t + t^2.$$

It follows that $\|a\|^2 \geq \alpha^2 + \beta^2 + 2\beta t$ and thus $\beta = 0$.

Now (3) follows easily, while (4) and (5) are trivial. □

Corollary 1.11. *Every unital algebraic homomorphism $\chi : A \to \mathbf{C}$ is a $*$-homomorphism.*

For a unital abelian C*-algebra A, a unital algebraic homomorphism $\chi : A \to \mathbf{C}$ is simply called a *character*. We will denote by $\Omega := \Omega(A)$ the set of characters of A. Sometimes Ω is called the *spectrum* of A. Observe that if $\chi : A \to \mathbf{C}$ is a character, we have that $\chi \in A^*$ and $\|\chi\|_{A^*} = 1$. One checks that Ω is closed for the $\sigma(A^*, A)$-topology and thus compact by Banach–Alaoglu Theorem. The *Gelfand Transform* $\gamma : A \to \mathrm{C}(\Omega)$ is defined by $\gamma(a)(\chi) = \chi(a)$.

Theorem 1.12. *Let A be any unital abelian* C**-algebra. Then the Gelfand Transform $\gamma : A \to \mathrm{C}(\Omega)$ is an onto $*$-isomorphism. Moreover $\sigma(a) = \{\chi(a) : \chi \in \Omega\}$ for all $a \in A$.*

Proof. Let $a \in A$. We have already shown that $\{\chi(a) : \chi \in \Omega\} \subset \sigma(a)$. If $\lambda \in \sigma(a)$, then $a - \lambda 1$ is not invertible. It is thus contained in a maximal proper ideal $\mathfrak{m}$, which is closed by Exercise 1.6. Observe that the Banach algebra $A/\mathfrak{m}$ is a division ring and so is isomorphic to $\mathbf{C}$. Whence there exists $\chi \in \Omega$ such that $\chi(a - \lambda 1) = 0$, that is, $\chi(a) = \lambda$. Therefore $\sigma(a) = \{\chi(a) : \chi \in \Omega\}$.

It is then clear that γ is a $*$-isomorphism and is isometric. Indeed, for all $a \in A$, since $a^*a = aa^*$, we have

$$\|\gamma(a)\|_\infty = \sup\{|\chi(a)| : \chi \in \Omega\} = r(a) = \|a\|.$$

Thus, $\gamma(A)$ is a unital closed $*$-subalgebra of $\mathrm{C}(\Omega)$. It remains to prove that γ is onto. Observe that $\gamma(A)$ separates points: for all $\chi \neq \chi'$, there exists $a \in A$ such that $\chi(a) \neq \chi'(a)$, that is, $\gamma(a)(\chi) \neq \gamma(a)(\chi')$. By Stone–Weierstrass's Theorem, $\gamma(A)$ is dense in $\mathrm{C}(\Omega)$. Therefore $\gamma(A) = \mathrm{C}(\Omega)$. □

Corollary 1.13. *If $a \in A$ is a unitary, then $\sigma(a) \subset \mathbf{T}$. If $a \in A$ is selfadjoint, then $\sigma(a) \subset \mathbf{R}$.*

Theorem 1.14 (Continuous functional calculus)**.** *Let A be a unital* C**-algebra and $b \in A$ be a normal element. Denote by B the abelian* C**-algebra generated by b. There exists a unique onto $*$-isomorphism $\Phi : \mathrm{C}(\sigma(b)) \to B$ such that $\Phi(z) = b$. We moreover have $\sigma(\Phi(f)) = f(\sigma(b))$.*

We will simply denote $\Phi(f)$ by $f(b)$. Observe that $\|f(b)\| \leq \|b\|\|f\|_\infty$.

Proof. Let Ω be the set of characters of B. Define the continuous function $\psi : \Omega \to \sigma(b)$ by $\psi(\chi) = \chi(b)$. We have seen before that ψ is onto. Assume now that $\psi(\chi) = \psi(\chi')$, that is, $\chi(b) = \chi'(b)$. It follows that $\chi(p(b, b^*)) = \chi'(p(b, b^*))$ for all polynomials p. Since b generates B, we get that $\chi = \chi'$. Therefore ψ is a homeomorphism. Then $\widehat{\psi} : \mathrm{C}(\Omega) \to \mathrm{C}(\sigma(b))$ defined by $\widehat{\psi}(f) = f \circ \psi$ is an onto $*$-isomorphism. Now the $*$-isomorphism $\Phi = \gamma^{-1} \circ \widehat{\psi}^{-1} : \mathrm{C}(\sigma(b)) \to B$ does the job. □

Exercise 1.15. Let A, B be any C*-algebras and $\pi : A \to B$ any injective $*$-homomorphism. Show that π is isometric.

1.1.3. *The Gelfand–Naimark–Segal construction.*

Definition 1.16. An element $a \in A$ is *positive* if $a = a^*$ and $\sigma(a) \subset \mathbf{R}_+$. We will denote $a \geq 0$. The set of positive elements in A will also be denoted by A_+.

An element $a \in A$ is *negative* if $-a$ is positive. The set of negative elements in A will be denoted by A_-. For selfadjoint elements $a, b \in A$, we write $a \leq b$ when $b - a \in A_+$.

Proposition 1.17. *Let A be a unital* C^**-algebra and let $a \in A$ be a selfadjoint element. There exists a unique pair (h, k) of positive elements in A such that $a = h - k$ and $hk = kh = 0$.*

Proof. Define the continuous functions $f(t) = \max(t, 0)$ and $g(t) = \max(-t, 0)$ so that $f(t) - g(t) = t$, $f(t) \geq 0$, $g(t) \geq 0$ and $f(t)g(t) = 0$. By continuous functional calculus, we have $a = f(a) - g(a)$, $f(a) \geq 0$, $g(a) \geq 0$ and $f(a)g(a) = g(a)f(a) = 0$. We have proven the existence of the decomposition. To prove the uniqueness, assume that $a = u - v$ for some $u, v \in A_+$ such that $uv = vu = 0$. It is not hard to see that u and v commute with a so that the C^*-algebra $\mathrm{C}^*(a, u, v)$ is abelian. There exists some compact space X such that $\mathrm{C}^*(a, u, v) = \mathrm{C}(X)$. Regarding a, u, v as continuous fonctions on X, it is clear that $u = \max(a, 0)$ and $v = \max(-a, 0)$. This implies that $u = h$ and $v = k$. □

Exercise 1.18. Let A be a unital C^*-algebra.

- Let $a \in A_+$ and $n \geq 1$. Show that there exists a unique $b \in A_+$ such that $a = b^n$. We then write $b := a^{1/n}$.
- Let $a \in A$ selfadjoint. Show that $a \geq 0$ if and only if $\|t - a\| \leq t$ for some $t \geq \|a\|$. Deduce that if $a, b \geq 0$, then $a + b \geq 0$.

Proposition 1.19. *Let A be a unital* C^**-algebra and $a \in A$. The following are equivalent:*

(1) $a \geq 0$.
(2) *There exists $b \in A$ such that $a = b^*b$.*

Proof. (1) ⇒ (2) It suffices to put $b = a^{1/2}$.

(2) ⇒ (1) Assume that $a = b^*b$ and write $a = h - k$ as in Proposition 1.17. We want to show that $k = 0$. Set $bk^{1/2} = \alpha + \mathrm{i}\beta$, with α, β selfadjoint elements in A. On the one hand, we have

$$(bk^{1/2})^*(bk^{1/2}) = k^{1/2}b^*bk^{1/2} = k^{1/2}(h - k)k^{1/2} = -k^2 \leq 0,$$

since $hk = kh = 0$. On the other hand,

$$(bk^{1/2})^*(bk^{1/2}) = (\alpha + \mathrm{i}\beta)^*(\alpha + \mathrm{i}\beta) = \alpha^2 + \beta^2 + \mathrm{i}(\alpha\beta - \beta\alpha).$$

Thus $\mathrm{i}(\alpha\beta - \beta\alpha) = -k^2 - \alpha^2 - \beta^2 \leq 0$. Observe that $\sigma((bk^{1/2})^*(bk^{1/2}))$ and $\sigma((bk^{1/2})(bk^{1/2})^*)$ only differ by 0 (see Exercise 1.5). Thus $(bk^{1/2})(bk^{1/2})^* = -c$ with $c \in A_+$. We get $-c = \alpha^2 + \beta^2 + \mathrm{i}(\beta\alpha - \alpha\beta)$, so that $\mathrm{i}(\alpha\beta - \beta\alpha) = c + \alpha^2 + \beta^2 \geq 0$. Therefore $\mathrm{i}(\alpha\beta - \beta\alpha) \in A_+ \cap A_-$ and so $\mathrm{i}(\alpha\beta - \beta\alpha) = 0$. This implies that $-k^2 = (bk^{1/2})^*(bk^{1/2}) = \alpha^2 + \beta^2 \in A_+ \cap A_-$ and thus $k = 0$. □

Exercise 1.20. Show that for all $a \in A$, $a^*a \leq \|a\|^2 1$.

Corollary 1.21. *Let A be any unital* C^**-algebra. Then A is linearly spanned by $\mathcal{U}(A)$.*

Proof. Up to considering real and imaginary parts and up to scaling, it suffices to show that any element $a \in (\Re(A))_1$ is a linear combination of unitaries. Indeed, since $a \in (\Re(A))_1$, we have $0 \leq a^2 \leq 1$. Put $u = a + \mathrm{i}\sqrt{1 - a^2}$. Then we have $u \in \mathcal{U}(A)$ and $a = \frac{1}{2}(u + u^*)$. □

Definition 1.22. A *state* $\varphi : A \to \mathbf{C}$ is a positive linear functional ($\varphi(a) \geq 0$ for all $a \geq 0$) such that $\varphi(1) = 1$. The state space of A is denoted by $\Sigma(A)$. A state φ is *faithful* if $\varphi(a^*a) > 0$ for all $a \neq 0$.

Example 1.23. Let (π, H, ξ) be a unital $*$-representation of A together with a unit vector. The linear functional $a \mapsto \langle \pi(a)\xi, \xi \rangle$ defines a state on A. We will prove that every state on a unital C^*-algebra arises this way.

Proposition 1.24. *Let $\varphi : A \to \mathbf{C}$ be a positive linear functional. The following hold true.*

1. *For all $a, b \in A$, $|\varphi(b^*a)|^2 \leq \varphi(a^*a)\varphi(b^*b)$*
2. *φ is bounded and $\|\varphi\| = \varphi(1)$. In particular, if φ is a state then $\|\varphi\| = 1$.*

Proof. Observe that $(a, b) \mapsto \varphi(b^*a)$ defines a semi-sesquilinear form on A. Then (1) follows from the Cauchy–Schwarz Inequality. For (2), observe that since $a^*a \leq \|a\|^2 1$, we have $|\varphi(a)|^2 \leq \varphi(1)\varphi(a^*a) \leq \varphi(1)^2\|a\|^2$. It follows that $\|\varphi\| = \varphi(1)$. □

Example 1.25. Let X be a compact space. Any probability measure μ on X gives rise to a state φ on $\mathrm{C}(X)$ by $\varphi(f) = \int_X f \, \mathrm{d}\mu$. By Riesz Representation Theorem, any state on $\mathrm{C}(X)$ arises this way.

Exercise 1.26. Let A be a unital C*-algebra and let $\varphi : A \to \mathbf{C}$ be a bounded linear functional with $\|\varphi\| = \varphi(1)$. Show that φ is positive. Deduce that if $B \subset A$ is a unital C*-subalgebra, then any state on B has an extension on A.

Theorem 1.27 (GNS construction)**.** *Let A be a unital* C**-algebra.*

1. *For every state φ on A, there exists a cyclic $*$-representation (π_φ, H_φ) together with a unit vector $\xi_\varphi \in H_\varphi$ such that $\varphi(a) = \langle \pi_\varphi(a)\xi_\varphi, \xi_\varphi\rangle$, for all $a \in A$.*
2. *If (π, H) is a cyclic $*$-representation with unit cyclic vector $\xi \in H$ and φ is the state defined by $\varphi(a) = \langle \pi(a)\xi, \xi\rangle$, then $\pi \cong \pi_\varphi$.*

Proof. (1) Let φ be a state on A. Define the following semi-sesquilinear form $\langle a, b\rangle_\varphi = \varphi(b^*a)$ on A. After separation and completion, promote $(A, \langle \cdot, \cdot\rangle_\varphi)$ to a genuine Hilbert space H_φ. Denote by $a^\bullet \in H_\varphi$ the image of $a \in A$ in H_φ. One checks that $\pi_\varphi(a)b^\bullet = (ab)^\bullet$ defines a cyclic $*$-representation with unit cyclic vector $\xi_\varphi = 1^\bullet$. Indeed, for all $a, b \in A$, we have

$$\begin{aligned}
\|\pi_\varphi(a)b^\bullet\|_\varphi^2 &= \langle \pi_\varphi(a)b^\bullet, \pi_\varphi(a)b^\bullet\rangle_\varphi \\
&= \langle \pi_\varphi(a^*a)b^\bullet, b^\bullet\rangle_\varphi \\
&= \varphi(b^*\, a^*a\, b) \\
&\leq \|a\|^2\varphi(b^*b) \\
&= \|a\|^2\|b^\bullet\|_\varphi^2
\end{aligned}$$

and hence $\pi_\varphi(a) \in \mathbf{B}(H_\varphi)$ is well-defined. For all $a \in A$, we moreover have

$$\langle \pi_\varphi(a)\xi_\varphi, \xi_\varphi\rangle_\varphi = \langle a^\bullet, 1^\bullet\rangle_\varphi = \varphi(a).$$

We leave (2) as an exercise. □

Corollary 1.28. *Every unital* C**-algebra admits a unital faithful $*$-representation (π, H). Moreover, H can be chosen to be separable if A is separable.*

Proof. Let $S \subset \Sigma(A)$ be a weak*-dense subset. Note that if A is separable, S can be taken countable. Define $\pi = \bigoplus_{\varphi \in S} \pi_\varphi$. Assume that $\pi(a) = 0$, that is, $\pi(a^*a) = 0$. We get $\varphi(a^*a) = 0$ for all $\varphi \in S$. By density, we get $\varphi(a^*a) = 0$ for all $\varphi \in \Sigma(A)$.

Let now μ be any Borel probability measure on $X := \sigma(a^*a)$ and define the state $\psi(f(a^*a)) = \int_X f \, \mathrm{d}\mu$ for all $f \in \mathrm{C}(X)$. Extend ψ to φ on A. We have

$$\int_X t \, \mathrm{d}\mu(t) = \psi(a^*a) = \varphi(a^*a) = 0.$$

It follows that $\mu(X \cap (0, +\infty)) = 0$. Since this holds true for any Borel probability measure on X, we have that $X = \{0\}$ and so $a = 0$. □

The above Corollary shows that the notions of selfadjoint, positive, unitary elements in a unital C*-algebra A correspond to the notions of selfadjoint, positive, unitary elements in $\mathbf{B}(H)$.

2. Spectral theorem

2.1. Topologies on $\mathbf{B}(H)$.

Definition 2.1. Let H be any complex Hilbert space.

- The *strong operator topology* (SOT) on $\mathbf{B}(H)$ is defined by the following family of open neighbourhoods: for $S \in \mathbf{B}(H)$, $\varepsilon > 0$, $\xi_1, \dots, \xi_n \in H$, define
$$\mathcal{U}(S, \varepsilon, \xi_i) := \{T \in \mathbf{B}(H) : \|(T - S)\xi_i\| < \varepsilon, \forall 1 \le i \le n\}.$$
- The *weak operator topology* (WOT) on $\mathbf{B}(H)$ is defined by the following family of open neighbourhoods: for $S \in \mathbf{B}(H)$, $\varepsilon > 0$, $\xi_1, \dots, \xi_n, \eta_1, \dots, \eta_n \in H$, define
$$\mathcal{V}(S, \varepsilon, \xi_i, \eta_i) := \{T \in \mathbf{B}(H) : |\langle (T - S)\xi_i, \eta_i\rangle| < \varepsilon, \forall 1 \le i \le n\}.$$

The strong operator topology is always stronger than the weak operator topology. It is strictly stronger when H is infinite dimensional.

Theorem 2.2. *Let $\mathcal{C} \subset \mathbf{B}(H)$ be a nonempty convex subset. Then the strong operator closure and the weak operator closure of $\mathcal{C}$ coincide.*

Proof. Assume T is in the weak operator closure of $\mathcal{C}$. Let $\xi_1, \dots, \xi_n \in H$. Let $K = H \oplus \cdots \oplus H$ be the n-fold direct sum of H with itself. Define the $*$-isomorphism $\rho : \mathbf{B}(H) \to \mathbf{B}(K)$ by $\rho(T)(\eta_1, \dots, \eta_n) = (T\eta_1, \dots, T\eta_n)$. Let $\xi = (\xi_1, \dots, \xi_n) \in K$. It is clear that $\rho(\mathcal{C})$ is a convex subset of $\mathbf{B}(K)$. Since T is in the weak operator closure of $\mathcal{C}$, $\rho(T)$ is in the weak operator closure of $\rho(\mathcal{C})$ and hence $\rho(T)\xi$ is in the weak closure of $\rho(\mathcal{C})\xi$. Since $\rho(\mathcal{C})\xi \subset K$ is convex, the Hahn–Banach Separation Theorem implies that $\rho(T)\xi$ is also in the norm closure of $\rho(\mathcal{C})\xi$. For $\varepsilon > 0$, there exists $S \in \mathcal{C}$ such that $\|S\xi_i - T\xi_i\| < \varepsilon$ for all $1 \le i \le n$. This shows that T is in the strong operator closure of $\mathcal{C}$. □

Proposition 2.3. *Let $V \subset \mathbf{B}(H)$ be any weakly closed subspace and $\varphi : V \to \mathbf{C}$ any linear functional. The following assertions are equivalent.*

1. *There exist $\xi_1, \dots, \xi_n, \eta_1, \dots, \eta_n \in H$ such that*
$$\varphi(T) = \sum_{i=1}^{n} \langle T\xi_i, \eta_i\rangle, \forall T \in V.$$
2. *φ is strongly continuous.*
3. *φ is weakly continuous.*

Proof. (1) $\Rightarrow$ (2) is clear. For (2) $\Rightarrow$ (1), let $\varepsilon > 0$ and $\xi_1, \dots, \xi_n \in H$ such that $|\varphi(x)| \le 1$ for all $x \in \mathcal{U}(0, \varepsilon, \xi_i)$. It follows that $|\varphi(x)| \le \frac{1}{\varepsilon}\sqrt{\sum_i \|x\xi_i\|^2}$ for all $x \in V$. Let $\xi = (\xi_1, \dots, \xi_n) \in \ell^2_n \otimes H$ and $\mathcal{K} = \overline{(1 \otimes V)\xi} \subset \ell^2_n \otimes H$. Define the continuous linear functional $\psi : \mathcal{K} \to \mathbf{C}$ by $\psi((1 \otimes x)\xi) = \varphi(x)$ for all $x \in V$. By Representation Theorem, there exists $\eta \in \mathcal{K}$ such that $\varphi(x) = \langle (1 \otimes x)\xi, \eta\rangle$ for all $x \in V$.

Observe that φ is continuous if and only if $\ker \varphi$ is closed. Since $\ker \varphi \subset \mathbf{B}(H)$ is a nonempty convex subset, the equivalence between (2) and (3) follows from Theorem 2.2. □

Theorem 2.4. *The unit ball $(\mathbf{B}(H))_1$ is weakly compact.*

Proof. Denote by $\mathbf{D}_{\xi,\eta}$ the closed unit disk in $\mathbf{C}$ of center 0 and radius $\|\xi\|\|\eta\|$. The map $(\mathbf{B}(H))_1 \ni T \mapsto (\langle T\xi,\eta\rangle)_{\xi,\eta\in H} \in \prod_{\xi,\eta\in H}\mathbf{D}_{\xi,\eta}$ is a homeomorphism from $(\mathbf{B}(H))_1$, endowed with the weak operator topology onto its image X. Note that $\prod_{\xi,\eta\in H}\mathbf{D}_{\xi,\eta}$ is compact for the product topology by Tychonoff's Theorem. It remains to show that the image X is closed.

Let $\alpha = (\alpha_{\xi,\eta}) \in \overline{X}$. There exists a net $(S_i)_{i\in I}$ of elements in $(\mathbf{B}(H))_1$ such that $\langle S_i\xi,\eta\rangle \to \alpha_{\xi,\eta}$, for all $\xi,\eta \in H$. We get that $H\times H \ni (\xi,\eta)\mapsto \alpha_{\xi,\eta}\in\mathbf{C}$ is a sesquilinear form such that $|\alpha_{\xi,\eta}| \leq \|\xi\|\|\eta\|$ for all $\xi,\eta\in H$. By Riesz Representation Theorem for sesquilinear forms, there exists $T\in(\mathbf{B}(H))_1$ such that $\alpha_{\xi,\eta} = \langle T\xi,\eta\rangle$, for all $\xi,\eta\in H$. □

Proposition 2.5. *Let $(T_i)_{i\in I}$ be an increasing net of selfadjoint operators such that $-C1 \leq T_i \leq C1$ for all $i\in I$. Then $(T_i)_{i\in I}$ has a limit with respect to the strong operator topology. Moreover, for all $S\in\mathbf{B}(H)$ such that $T_i\leq S$ for all $i\in I$, we have that $\lim T_i \leq S$. We denote $\lim T_i = \sup T_i$.*

Proof. By weak compactness of the unit ball, we can find a subnet $(T_j)_{j\in J}$ which converges weakly to some selfadjoint operator $T\in\mathbf{B}(H)$.

Let $i\in I$. For all $j\geq i$, $\xi\in H$, we have $\langle T_j\xi,\xi\rangle \geq \langle T_i\xi,\xi\rangle$ so that $\langle T\xi,\xi\rangle = \lim_j\langle T_j\xi,\xi\rangle \geq \langle T_i\xi,\xi\rangle$. Thus, for all $i\geq j$, we have $0\leq T-T_i\leq T-T_j$ so that

$$\|(T-T_i)^{1/2}\xi\|^2 = \langle (T-T_i)\xi,\xi\rangle \leq \langle (T-T_j)\xi,\xi\rangle \to 0 \text{ as } j\to\infty.$$

We have that $(T-T_i)^{1/2}\to 0$ strongly as $i\to\infty$. Finally, strong continuity of multiplication on uniformly bounded sets implies that $(T-T_i)\to 0$ strongly as $i\to\infty$.

We have already seen that $T_i\leq T$ for all $i\in I$. Assume now that $T_i\leq S$ for all $i\in I$. Since $T_i\to T$ strongly as $i\to\infty$, we have that $T_i\to T$ weakly as $i\to\infty$, whence for all $\xi\in H$, we have $\langle T\xi,\xi\rangle = \lim_i\langle T_i\xi,\xi\rangle \leq \langle S\xi,\xi\rangle$. □

Definition 2.6. Let H be any complex Hilbert space.

- The *ultrastrong operator topology* on $\mathbf{B}(H)$ is defined by the following family of open neighbourhoods: for $S\in\mathbf{B}(H)$, $\varepsilon>0$, $(\xi_n)_n\in\ell^2(\mathbf{N},H)$, define
$$\mathcal{U}(S,\varepsilon,(\xi_n)_n) := \left\{T\in\mathbf{B}(H) : \sum_n \|(T-S)\xi_n\|^2 < \varepsilon\right\}.$$
- The *ultraweak operator topology* on $\mathbf{B}(H)$ is defined by the following family of open neighbourhoods: for $S\in\mathbf{B}(H)$, $\varepsilon>0$, $(\xi_n)_n,(\eta_n)_n\in\ell^2(\mathbf{N},H)$, define
$$\mathcal{V}(S,\varepsilon,(\xi_n)_n,(\eta_n)_n) := \left\{T\in\mathbf{B}(H) : \left|\sum\langle (T-S)\xi_n,\eta_n\rangle\right| < \varepsilon\right\}.$$

Observe that the ultrastrong (resp. ultraweak) operator topology on $\mathbf{B}(H)$ correspond to the pullback of the strong (resp. weak) operator topology on $\mathbf{B}(\ell^2(\mathbf{N},H))$ under the map $\pi : \mathbf{B}(H)\to\mathbf{B}(\ell^2(\mathbf{N},H)) : T\mapsto ((\xi_n)_n\mapsto(T\xi_n)_n)$.

Exercise 2.7. Show that on uniformly bounded sets, weak and ultraweak (resp. strong and ultrastrong) operator topologies coincide.

Proposition 2.8. *Let $V\subset\mathbf{B}(H)$ be any ultraweakly closed subspace and $\varphi : V\to\mathbf{C}$ any linear form. The following are equivalent.*

(1) *There exist $(\xi_n)_n,(\eta_n)_n\in\ell^2(\mathbf{N},H)$ such that*
$$\varphi(T) = \sum_n\langle T\xi_n,\eta_n\rangle, \forall T\in V.$$

(2) φ *is ultrastrongly continuous.*
(3) φ *is ultraweakly continuous.*
(4) φ *is strongly continuous on* $(V)_1$.
(5) φ *is weakly continuous on* $(V)_1$.

Proof. The proof is analogous to Proposition 2.3, so we leave it as an exercise. $\square$

2.2. **Spectral measures.**

Definition 2.9. Let H be any complex Hilbert space and (X, Ω) any standard Borel space together with its σ-algebra of Borel subsets. A *spectral measure* for (X, Ω, H) is a function $\Phi : \Omega \to \mathbf{B}(H)$ which satisfies the following properties:

(1) $\Phi(\mathcal{U})$ is a projection, that is, $\Phi(\mathcal{U}) = \Phi(\mathcal{U})^* = \Phi(\mathcal{U})^2$ for all $\mathcal{U} \in \Omega$.
(2) $\Phi(\emptyset) = 0$ and $\Phi(X) = 1$.
(3) $\Phi(\mathcal{U} \cap \mathcal{V}) = \Phi(\mathcal{U})\Phi(\mathcal{V})$ for all $\mathcal{U}, \mathcal{V} \in \Omega$.
(4) Whenever $(\mathcal{U}_n)_n$ is a sequence of pairwise disjoint Borel subsets of X, we have

$$\Phi\left(\bigcup_n \mathcal{U}_n\right) = \sum_n \Phi(\mathcal{U}_n).$$

The above convergence holds with respect to the strong operator topology.

Example 2.10. Let (X, Ω, μ) be any standard Borel probability space. Regard $\mathrm{L}^\infty(X, \mu) \subset \mathbf{B}(\mathrm{L}^2(X, \mu))$ where $\mathrm{L}^\infty(X, \mu)$ acts by multiplication. Then the map $\Phi : \Omega \to \mathrm{L}^\infty(X, \mu)$ defined by $\Phi(\mathcal{U}) = \mathbf{1}_\mathcal{U}$ is a spectral measure for $(X, \Omega, \mathrm{L}^2(X, \mu))$.

The next lemma will be useful. The proof is left to the reader.

Lemma 2.11. *Let Φ be any spectral measure for (X, Ω, H). Let $\xi, \eta \in H$. Then the map $\Phi_{\xi,\eta} : \Omega \to \mathbf{C} : \mathcal{U} \mapsto \langle \Phi(\mathcal{U})\xi, \eta\rangle$ defines a Borel complex measure on X with $\|\Phi_{\xi,\eta}\| \leq \|\xi\|\|\eta\|$. In particular, $\Phi_{\xi,\xi}$ is a Borel probability measure on X for every $\xi \in H$ such that $\|\xi\| = 1$.*

Denote by $\mathrm{B}(X)$ the C^*-algebra of all bounded Borel functions on X.

Proposition 2.12. *Let Φ be any spectral measure for (X, Ω, H) and $f \in \mathrm{B}(X)$. Then there exists a unique operator $T \in \mathbf{B}(H)$ which satisfies the following property: for every $\varepsilon > 0$ and every Ω-partition $(\mathcal{U}_1, \dots, \mathcal{U}_n)$ of X such that $\sup\{|f(x) - f(y)| : 1 \leq k \leq n, x, y \in \mathcal{U}_k\} \leq \varepsilon$ and for every $x_k \in \mathcal{U}_k$, we have*

$$\left\| T - \sum_{k=1}^n f(x_k)\Phi(\mathcal{U}_k) \right\|_\infty \leq \varepsilon.$$

Proof. Define the sesquilinear form on $H \times H$ by $\varphi(\xi, \eta) = \int_X f \, \mathrm{d}\Phi_{\xi,\eta}$. We have $|\varphi(\xi, \eta)| \leq \|f\|_\infty \|\xi\|\|\eta\|$ by Lemma 2.11 and hence φ is bounded. By Riesz Representation Theorem, there exists a unique operator $T \in \mathbf{B}(H)$ such that $\langle T\xi, \eta\rangle = \varphi(\xi, \eta) = \int_X f \, \mathrm{d}\Phi_{\xi,\eta}$ for all $\xi, \eta \in H$.

Let now $\varepsilon > 0$ and $x_k \in \mathcal{U}_k$ for every $1 \leq k \leq n$ as in the statement. We have

$$\begin{aligned}\left|\left\langle (T - \sum_{k=1}^{n} f(x_k)\Phi(\mathcal{U}_k))\xi, \eta \right\rangle\right| &= \left|\sum_{k=1}^{n} \int_{\mathcal{U}_k} (f(x) - f(x_k))\, \mathrm{d}\Phi_{\xi,\eta}(x)\right| \\ &\leq \sum_{k=1}^{n} \int_{\mathcal{U}_k} |f(x) - f(x_k)|\, \mathrm{d}|\Phi_{\xi,\eta}|(x) \\ &\leq \varepsilon \int_X \mathrm{d}|\Phi_{\xi,\eta}|(x) \\ &\leq \varepsilon \|\xi\| \|\eta\|.\end{aligned}$$

This implies the inequality in the statement. □

The operator T will be denoted by $\int_X f \, \mathrm{d}\Phi$. We have

$$\forall \xi, \eta \in H, \quad \left\langle \left(\int_X f \, \mathrm{d}\Phi \right) \xi, \eta \right\rangle = \int_X f \, \mathrm{d}\Phi_{\xi,\eta}.$$

The proof of the next proposition is left as an exercise.

Proposition 2.13. *Let Φ be any spectral measure for (X, Ω, H). The map $\pi : \mathrm{B}(X) \to \mathbf{B}(H)$ defined by $\pi(f) = \int_X f \, \mathrm{d}\Phi$ is a unital $*$-representation. Moreover $\pi(f)$ is a normal operator for every $f \in \mathrm{B}(X)$.*

Let X be any compact space. Let $\mathcal{M}(X)$ be the Banach space of all finite Borel measures on X. By Riesz Representation Theorem, we have $\mathrm{C}(X)^* = \mathcal{M}(X)$. Identify $\mathrm{B}(X)$ as a subspace of $\mathcal{M}(X)^*$ in the following way: for every $f \in \mathrm{B}(X)$, we have $\mu \mapsto \int_X f \, \mathrm{d}\mu \in \mathcal{M}(X)^*$ and $\|f\|_\infty = \|\mu \mapsto \int_X f \, \mathrm{d}\mu\|$. Since $(\mathrm{C}(X))_1$ is weak*-dense in $(\mathrm{C}(X)^{**})_1 = (\mathcal{M}(X)^*)_1$, it follows that for every $f \in \mathrm{B}(X) \subset \mathcal{M}(X)^*$, there exists a net $(f_i)_{i \in I}$ in $\mathrm{C}(X)$ such that $\|f_i\|_\infty \leq \|f\|_\infty$ for every $i \in I$ and $\lim_i \int_X f_i \, \mathrm{d}\mu = \int_X f \, \mathrm{d}\mu$ for every $\mu \in \mathcal{M}(X)$.

Theorem 2.14. *Let X be any compact space and $\pi : \mathrm{C}(X) \to \mathbf{B}(H)$ any unital $*$-representation. Then there exists a unique spectral measure Φ for (X, Ω, H) such that $\Phi_{\xi,\eta}$ is a regular Borel complex measure on X and $\pi(f) = \int_X f \, \mathrm{d}\Phi$ for every $f \in \mathrm{C}(X)$.*

Proof. Let $\xi, \eta \in H$. The map $\mathrm{C}(X) \to \mathbf{C} : f \mapsto \langle \pi(f)\xi, \eta \rangle$ defines a bounded linear functional with norm at most $\|\xi\| \|\eta\|$. By Riesz Representation Theorem, there exists a unique Borel complex measure $\mu_{\xi,\eta} \in \mathcal{M}(X)$ such that $\|\mu_{\xi,\eta}\| \leq \|\xi\| \|\eta\|$ and $\langle \pi(f)\xi, \eta \rangle = \int_X f \, \mathrm{d}\mu_{\xi,\eta}$ for every $f \in \mathrm{C}(X)$.

Let $f \in \mathrm{B}(X)$. The sesquilinear form defined by $\varphi(\xi, \eta) = \int_X f \, \mathrm{d}\mu_{\xi,\eta}$ is bounded by $\|f\|_\infty$. By Riesz Representation Theorem, there is a unique bounded operator $\widetilde{\pi}(f) \in \mathbf{B}(H)$ such that $\langle \widetilde{\pi}(f)\xi, \eta \rangle = \int_X f \, \mathrm{d}\mu_{\xi,\eta}$ for all $\xi, \eta \in H$.

Claim 2.15. The map $\widetilde{\pi} : \mathrm{B}(X) \to \mathbf{B}(H)$ is a unital $*$-representation such that $\widetilde{\pi}(f) = \pi(f)$ for every $f \in \mathrm{C}(X)$.

The fact that $\widetilde{\pi}(f) = \pi(f)$ for every $f \in \mathrm{C}(X)$ is clear from the definitions. We only prove that $\widetilde{\pi}$ is multiplicative. Let $f \in \mathrm{B}(X)$, $g \in \mathrm{C}(X)$ and let $(f_i)_{i \in I}$ be a net in $\mathrm{C}(X)$ such that $\|f_i\|_\infty \leq \|f\|_\infty$ for every $i \in I$ and $f_i \to f$ with respect to the weak*-topology as $i \to \infty$. We have $\widetilde{\pi}(f_i) \to \widetilde{\pi}(f)$ weakly as $i \to \infty$. Regarding $g\mu_{\xi,\eta}$ as an element in $\mathcal{M}(X)$, we have $\widetilde{\pi}(f_i g) \to \widetilde{\pi}(fg)$ weakly as well. This yields

$$\widetilde{\pi}(fg) = \lim_i \widetilde{\pi}(f_i g) = \lim_i \pi(f_i g) = (\lim_i \pi(f_i))\pi(g) = \widetilde{\pi}(f)\widetilde{\pi}(g).$$

Repeating the same reasoning with $g \in \mathrm{B}(X)$ proves that $\widetilde{\pi}$ is indeed multiplicative. This finishes the proof of the Claim.

Define now $\Phi(\mathcal{U}) = \widetilde{\pi}(\mathbf{1}_{\mathcal{U}})$ for every $\mathcal{U} \in \Omega$. It is easy to check that Φ is a spectral measure for (X, Ω, H) for which $\widetilde{\pi}(f) = \int_X f \, \mathrm{d}\Phi$ for every $f \in \mathrm{B}(X)$. In particular, we have $\pi(f) = \int_X f \, \mathrm{d}\Phi$ for every $f \in \mathrm{C}(X)$. □

2.3. **Spectral Theorem.**

Theorem 2.16 (Fuglede Theorem)**.** *Let $T \in \mathbf{B}(H)$ be any normal operator and $S \in \mathbf{B}(H)$. If $TS = ST$, then $T^* S = ST^*$.*

Proof. By continuous functional calculus, we get $\exp(\mathrm{i}\overline{z}T)S = S\exp(\mathrm{i}\overline{z}T)$ for every $z \in \mathbf{C}$. Define the entire analytic function $f : \mathbf{C} \to \mathbf{B}(H)$ by

$$\begin{aligned} f(z) &= \exp(-\mathrm{i}zT^*)S\exp(\mathrm{i}zT^*) \\ &= \exp(-\mathrm{i}zT^*)\exp(-\mathrm{i}\overline{z}T)S\exp(\mathrm{i}\overline{z}T)\exp(\mathrm{i}zT^*) \\ &= \exp(-\mathrm{i}(zT^* + \overline{z}T))S\exp(\mathrm{i}(\overline{z}T + zT^*)) \end{aligned}$$

since T and T^* commute. Observe that since $zT^* + \overline{z}T$ is selfadjoint, $\exp(\mathrm{i}(zT^* + \overline{z}T))$ is a unitary and thus $f(z)$ is uniformly bounded. By Liouville's Theorem and Hahn-Banach Theorem, f is a constant function and hence $f' = 0$. Therefore $0 = f'(z) = -\mathrm{i}T^*f(z) + \mathrm{i}f(z)T^*$ for every $z \in \mathbf{C}$. With $z = 0$, we get $0 = -\mathrm{i}T^*S + \mathrm{i}ST^*$. □

Let $T \in \mathbf{B}(H)$ be any normal operator. The continuous functional calculus gives rise to a unital $*$-representation $\pi : \mathrm{C}(\sigma(T)) \to \mathrm{C}^*(T) \subset \mathbf{B}(H)$ where $\pi(z) = T$. We will simply denote $\pi(f) = f(T)$.

Theorem 2.17 (Spectral Theorem)**.** *Let $T \in \mathbf{B}(H)$ be any normal operator. Then there exists a unique spectral measure Φ for $(\sigma(T), \Omega, H)$ such that the following assertions hold:*

1. *$f(T) = \int_{\sigma(T)} f \, \mathrm{d}\Phi$ for every $f \in \mathrm{C}(\sigma(T))$.*
2. *If $\mathcal{U} \subset \sigma(T)$ is a nonempty open subset, then $\Phi(\mathcal{U}) \neq 0$.*
3. *For every $S \in \mathbf{B}(H)$, $ST = TS$ if and only if $S\Phi(\mathcal{U}) = \Phi(\mathcal{U})S$ for every $\mathcal{U} \in \Omega$.*

Proof. The existence of Φ has already been proven in Theorem 2.14. To prove (2), choose a nonzero continuous function $f \in \mathrm{C}(\sigma(T))$ such that $0 \leq f \leq \mathbf{1}_{\mathcal{U}}$. We have $0 \neq \pi(f) \leq \widetilde{\pi}(\mathbf{1}_{\mathcal{U}}) = \Phi(\mathcal{U})$.

To prove (3) first assume that $ST = TS$. By Fuglede Theorem, we have $ST^* = T^*S$ as well. By continuous functional calculus, we get $S\pi(f) = \pi(f)S$ for every $f \in \mathrm{C}(\sigma(T))$. Now given $f \in \mathrm{B}(\sigma(T))$, let $(f_i)_{i \in I}$ be a net in $\mathrm{C}(\sigma(T))$ such that $\|f_i\|_\infty \leq \|f\|_\infty$ for every $i \in I$ and $f_i \to f$ with respect to the weak* topology as $i \to \infty$. It follows that $\pi(f_i) \to \widetilde{\pi}(f)$ weakly as $i \to \infty$. Therefore $S\widetilde{\pi}(f) = \widetilde{\pi}(f)S$ for every $f \in \mathrm{B}(\sigma(T))$. In particular, $S\Phi(\mathcal{U}) = \Phi(\mathcal{U})S$ for every $\mathcal{U} \in \Omega$. It is easy to check that if $S\Phi(\mathcal{U}) = \Phi(\mathcal{U})S$ for every $\mathcal{U} \in \Omega$ then $S\widetilde{\pi}(f) = \widetilde{\pi}(f)S$ for every $f \in \mathrm{B}(\sigma(T))$. This finishes the proof. □

We will simply denote $\widetilde{\pi}(f) = f(T)$ for every $f \in \mathrm{B}(\sigma(T))$. The unital $*$-representation $\mathrm{B}(\sigma(T)) \to \mathbf{B}(H) : f \mapsto f(T)$ is called the *Borel functional calculus*.

Theorem 2.18. *Let $T \in \mathbf{B}(H)$ be any normal operator together with its spectral measure defined for $(\sigma(T), \Omega, H)$. The unital $*$-representation $\mathrm{B}(\sigma(T)) \to \mathbf{B}(H) : f \mapsto f(T)$ satisfies the following property: whenever $(f_i)_{i \in I}$ is a net in $\mathrm{B}(\sigma(T))$ such that $f_i \to 0$ weak* as $i \to \infty$, we have $f_i(T) \to 0$ weakly as $i \to \infty$.*

Moreover, if $\rho : \mathrm{B}(\sigma(T)) \to \mathbf{B}(H)$ is another unital $*$-representation such that $\rho(z) = T$ and $\rho(f_i) \to 0$ weakly as $i \to \infty$ whenever $f_i \to 0$ weak as $i \to \infty$, then $\rho(f) = f(T)$ for every $f \in \mathrm{B}(\sigma(T))$.*

3. Introduction to von Neumann algebras

3.1. **Definition and first examples of von Neumann algebras.** For any nonempty subset $\mathcal{S} \subset \mathbf{B}(H)$, the *commutant* of $\mathcal{S}$ is defined by

$$\mathcal{S}' := \{T \in \mathbf{B}(H) : ST = TS, \forall S \in \mathcal{S}\}.$$

It is easy to see that one always has $\mathcal{S} \subset \mathcal{S}''$. Moreover, if $\mathcal{S}$ is stable under the adjoint operation, then $\mathcal{S}' \subset \mathbf{B}(H)$ is a weakly closed unital $*$-subalgebra.

Theorem 3.1 (Bicommutant Theorem)**.** *Let $M \subset \mathbf{B}(H)$ be any unital $*$-subalgebra. The following assertions are equivalent.*

1. *$M = M''$.*
2. *M is strongly closed.*
3. *M is weakly closed.*
4. *M is ultrastrongly closed.*
5. *M is ultraweakly closed.*

Proof. (1) $\Rightarrow$ (2). Let $(x_i)_{i \in I}$ be a net in M such that $x_i \to x$ strongly as $i \to \infty$. Since $x_i T = T x_i$ for all $i \in I$ and $T \in M'$, by passing to the limit we get $xT = Tx$, for all $T \in M'$. Thus $x \in M$. (2) $\Rightarrow$ (4) is obvious.

(4) $\Rightarrow$ (1). Let $x \in M''$ and $(\xi_n)_n \in \ell^2(\mathbf{N}, H)$. Let

$$\mathcal{U}(x, \varepsilon, (\xi_n)_n) := \left\{ y \in \mathbf{B}(H) : \sum_{n \in \mathbf{N}} \|(x - y)\xi_n\|^2 < \varepsilon^2 \right\}$$

be an ultrastrong neighborhood of x in $\mathbf{B}(H)$. Let $K = \ell^2(\mathbf{N}, H)$ and define $\rho : \mathbf{B}(H) \to \mathbf{B}(\ell^2(\mathbf{N}, H)) : T \mapsto 1_{\ell^2(\mathbf{N})} \otimes T$. Let $\xi = (\xi_n)_n \in K$. Define $V = \overline{\rho(M)\xi} \subset K$. Denote by $P_V : K \to V \in \mathbf{B}(K)$ the corresponding orthogonal projection. We have $\rho(a)P_V = P_V \rho(a)$ for all $a \in M$ and hence $P_V \in \rho(M)'$. Observe that $\rho(M)'$ can be identified inside $\mathbf{B}(\ell^2(\mathbf{N}, H))$ as the set of infinite matrices indexed by $\mathbf{N} \times \mathbf{N}$ with coefficients in M'. Since $x \in (M')'$, it follows that $\rho(x)P_V = P_V \rho(x)$. Thus $\rho(x)\xi \in V$ and hence we can find $y \in M$ such that $\|(\rho(x) - \rho(y))\xi\| < \varepsilon$. In particular, we have that $y \in \mathcal{U}(x, \varepsilon, (\xi_n)_n)$. Then M'' is contained in the ultrastrong closure of M and hence $M = M''$.

Since $M \subset \mathbf{B}(H)$ is convex, (2) $\Leftrightarrow$ (3) follows from Theorem 2.2. Likewise, (4) $\Leftrightarrow$ (5). □

Definition 3.2. A *von Neumann algebra* M is a unital $*$-subalgebra of $\mathbf{B}(H)$ which satisfies one of the equivalent conditions of Theorem 3.1.

The first important example of von Neumann algebras we discuss comes from *measure theory.* Let (X, μ) be a standard probability space. Define the unital $*$-representation $\pi : \mathrm{L}^\infty(X, \mu) \to \mathbf{B}(\mathrm{L}^2(X, \mu))$ given by multiplication: $(\pi(f)\xi)(x) = f(x)\xi(x)$ for all $f \in \mathrm{L}^\infty(X, \mu)$ and all

$\xi \in \mathrm{L}^2(X, \mu)$. Since π is a C*-algebraic isometric isomorphism, we will identify $f \in \mathrm{L}^\infty(X, \mu)$ with its image $\pi(f) \in \mathbf{B}(\mathrm{L}^2(X, \mu))$. When no confusion is possible, we will simply denote $\mathrm{L}^\infty(X, \mu)$ by $\mathrm{L}^\infty(X)$.

Proposition 3.3. *We have $\mathrm{L}^\infty(X)' \cap \mathbf{B}(\mathrm{L}^2(X, \mu)) = \mathrm{L}^\infty(X)$, that is, $\mathrm{L}^\infty(X)$ is maximal abelian in $\mathbf{B}(\mathrm{L}^2(X, \mu))$. In particular, $\mathrm{L}^\infty(X)$ is a von Neumann algebra.*

Proof. Let $T \in \mathrm{L}^\infty(X)' \cap \mathbf{B}(\mathrm{L}^2(X, \mu))$ and denote $f = T\mathbf{1}_X \in \mathrm{L}^2(X, \mu)$. For all $\xi \in \mathrm{L}^\infty(X) \subset \mathrm{L}^2(X, \mu)$, we have

$$T\xi = T\xi\,\mathbf{1}_X = \xi T\,\mathbf{1}_X = \xi f = f\xi.$$

For every $n \geq 1$, put $\mathcal{U}_n := \left\{x \in X : |f(x)| \geq \|T\|_\infty + \frac{1}{n}\right\}$. We have

$$\left(\|T\|_\infty + \frac{1}{n}\right) \mu(\mathcal{U}_n)^{1/2} \leq \|f\mathbf{1}_{\mathcal{U}_n}\|_2 = \|T\mathbf{1}_{\mathcal{U}_n}\|_2 \leq \|T\|_\infty\, \mu(\mathcal{U}_n)^{1/2},$$

hence $\mu(\mathcal{U}_n) = 0$ for every $n \geq 1$. This implies that $\|f\|_\infty \leq \|T\|_\infty$ and so $T = f$. □

The von Neumann algebra $M = \mathrm{L}^\infty(X)$ comes equipped with the faithful trace τ_μ given by integration against the probability measure μ,

$$\tau_\mu(f) = \int_X f\, \mathrm{d}\mu, \forall f \in \mathrm{L}^\infty(X).$$

Theorem 3.4 (Borel functional calculus in von Neumann algebras)**.** *Let H be any separable Hilbert space and $T \in \mathbf{B}(H)$ any normal operator. Denote by $A_T = \{T, T^*\}''$ the abelian von Neumann subalgebra generated by T and T^*.*

Then the map $\mathrm{B}(\sigma(T)) \to A_T : f \mapsto f(T)$ is an onto $*$-homomorphism. Moreover, there exists a Borel probability measure μ_T on $\mathrm{Sp}(\sigma(T))$ such that $\mathrm{L}^\infty(\sigma(T), \mu_T) \cong A_T$.

Proof. Let $S \in \{T, T^*\}' \cap \mathbf{B}(H)$. Then for every $f \in \mathrm{B}(\sigma(T))$, we have $Sf(T) = f(T)S$ by Theorem 2.17 (3). This implies that $f(T) \in \{T, T^*\}'' = A_T$. Observe that $\mathrm{C}(\sigma(T))$ is weak*-dense in $\mathrm{B}(\sigma(T))$ and $\mathrm{C}^*(T, T^*)$ is weakly dense in A_T. Since the map $\mathrm{B}(\sigma(T)) \to A_T : f \mapsto f(T)$ is weak*-weak continuous, it follows that it is onto.

By Zorn's Lemma, there exists a maximal family $(\xi_i)_{i \in I}$ of pairwise orthogonal unit vectors in H such that $H = \bigoplus_{i \in I} \overline{A_T \xi_i}$. Since H is separable, I is at most countable. Choose a sequence of positive reals $(\alpha_i)_{i \in I}$ such that $\sum_{i \in I} \alpha_i^2 = 1$. Put $\xi = \sum_{i \in I} \alpha_i \xi_i \in H$ and $\mu_T(\mathcal{U}) = \langle \mathbf{1}_{\mathcal{U}}(T)\xi, \xi\rangle$ for every $\mathcal{U} \subset \sigma(T)$ Borel subset. Then μ_T is a Borel probability measure on $\sigma(T)$. For every $f \in \mathrm{B}(\sigma(T))$, we have

$$f(T) = 0 \quad \text{if and only if} \quad f(T)\xi = 0 \quad \text{if and only if} \quad f = 0 \;\; \mu_T\text{-almost everywhere.}$$

Thus, $\ker\left(\mathrm{B}(\sigma(T)) \to A_T : f \mapsto f(T)\right) = \{f \in \mathrm{B}(\sigma(T)) : f = 0 \; \mu_T\text{-almost everywhere}\}$ and hence $\mathrm{L}^\infty(\sigma(T), \mu_T) \cong A_T$. □

Observe that for any von Neumann M, the center of M defined by $\mathcal{Z}(M) = M' \cap M$ is an abelian von Neumann algebra.

Definition 3.5. Let $M \subset \mathbf{B}(H)$ be a von Neumann algebra. We say that

- $p \in M$ is a *projection* if $p = p^* = p^2$.
- $v \in M$ is an *isometry* if $v^*v = 1$.
- $u \in M$ is a *partial isometry* if u^*u is a projection.

Observe that if u^*u is a projection, then uu^* is a projection as well. The set of projections of M will be denoted by $\mathcal{P}(M)$. If $K \subset H$ is a closed subspace, we denote by $[K] \in \mathbf{B}(H)$ the orthogonal projection $[K] : H \to K$.

We will always assume that M is *σ-finite*, that is, any family $(p_i)_{i\in I}$ of pairwise orthogonal projections in M is (at most) countable.

Exercise 3.6. Let M be any von Neumann algebra. The closed subspace $K \subset H$ is u-invariant for all $u \in \mathcal{U}(M)$ if and only if $[K] \in M'$.

Exercise 3.7. Let M be any von Neumann algebra and $\mathcal{I} \subset M$ any ultraweakly closed two-sided $*$-ideal. Show that there exists a central projection $z \in \mathcal{Z}(M)$ such that $\mathcal{I} = Mz$.

If $(p_i)_{i\in I}$ is a family of projections, we denote by

$$\bigvee_{i\in I} p_i = \left[\overline{\sum_{i\in I} \operatorname{ran}(p_i)}\right] \quad \text{and} \quad \bigwedge_{i\in I} p_i = \left[\bigcap_{i\in I} \operatorname{ran}(p_i)\right].$$

If $p \in \mathbf{B}(H)$ is a projection, write $p^\perp = 1 - p$. It is easy to check that $(\bigvee_{i\in I} p_i)^\perp = \bigwedge_{i\in I} p_i^\perp$.

Proposition 3.8. *Let $M \subset \mathbf{B}(H)$ be a von Neumann algebra. Then $\mathcal{P}(M)$ is a complete lattice.*

Proof. Let $(p_i)_{i\in I}$ be a family of projections in M. Since $M = (M')'$, we have that $\operatorname{ran}(p_i)$ is u-invariant for all $u \in \mathcal{U}(M')$ and all $i \in I$. Thus $\overline{\sum_{i\in I} \operatorname{ran}(p_i)}$ is u-invariant for all $u \in \mathcal{U}(M')$, whence $\bigvee_{i\in I} p_i \in M$. Moreover $\bigwedge_{i\in I} p_i = (\bigvee p_i^\perp)^\perp \in M$. □

Theorem 3.9 (Polar decomposition)**.** *Let $M \subset \mathbf{B}(H)$ be any von Neumann algebra and $T \in M$ any element. Then T can be written $T = U|T|$ where $|T| \in M$ and $U \in M$ is a partial isometry with initial support $\overline{\operatorname{ran}(T^*)}$ and final support $\overline{\operatorname{ran}(T)}$.*

Moreover, if $T = VS$ with $S \geq 0$ and V a partial isometry such that $V^*V = [\overline{\operatorname{ran}(S)}]$, then $S = |T|$ and $V = U$.

Proof. Since $T \in M$, we have $|T| = (T^*T)^{1/2} \in M$. Observe that $\ker(T) = \ker(T^*T) = \ker(|T|)$ so that $\overline{\operatorname{ran}(T^*)} = \ker(T)^\perp = \ker(|T|)^\perp = \overline{\operatorname{ran}(|T|)}$. Define $U\eta = 0$ for $\eta \in \operatorname{ran}(|T|)^\perp$ and $U|T|\xi = T\xi$ for all $\xi \in H$. One checks that $U \in \mathbf{B}(H)$ is a well-defined partial isometry such that $U^*U = [\overline{\operatorname{ran}(T^*)}]$, $UU^* = [\overline{\operatorname{ran}(T)}]$ and $T = U|T|$.

Assume now that $T = VS$ with $S \geq 0$ and $V^*V = [\overline{\operatorname{ran}(S)}]$. Then $T^*T = SV^*VS = S^2$. Thus $S = (T^*T)^{1/2} = |T|$. The formula $T = V|T|$ clearly shows that $V = U$.

Finally, using uniqueness, we can prove that $U \in M$. Indeed, let $v \in \mathcal{U}(M')$ be any unitary. Then $vTv^* = vUv^*\, v|T|v^*$. Since $T = vTv^*$, we obtain $|T| = v|T|v^*$. Since $U^*U = [\overline{\operatorname{ran}(|T|)}] \in M$, we have $(vUv^*)^*(vUv^*) = vU^*Uv^* = U^*U$ and hence $U = vUv^*$ by uniqueness. This implies that $U \in (M')' = M$. □

3.2. **The predual.** Let M be any von Neumann algebra. Denote by $M_* \subset M^*$ the subspace of all ultraweakly continuous functionals on M. Recall the following fact.

Proposition 3.10. *We have that M_* is a closed subspace of M^*. Therefore, $(M_*, \|\cdot\|)$ is a Banach space.*

Proof. Let $\varphi \in M^*$ and $(\varphi_i)_{i\in I}$ be a net in M_* such that $\lim \|\varphi - \varphi_i\| = 0$. We have to show that φ is strongly continuous on $(M)_1$. Let $x \in (M)_1$ and $(x_j)_{j\in J}$ a net in $(M)_1$ such that $x_j \to x$ strongly as $j \to \infty$. We have

$$\begin{aligned}|\varphi(x) - \varphi(x_j)| &\leq |\varphi(x) - \varphi_i(x)| + |\varphi_i(x) - \varphi_i(x_j)| + |\varphi_i(x_j) - \varphi(x_j)| \\ &\leq 2\|\varphi - \varphi_i\| + |\varphi_i(x) - \varphi_i(x_j)|.\end{aligned}$$

Let $\varepsilon > 0$. Choose $i \in I$ such that $\|\varphi - \varphi_i\| \leq \varepsilon/3$. Since φ_i is ultraweakly continuous, choose $j_0 \in J$ such that for all $j \geq j_0$, $|\varphi_i(x) - \varphi_i(x_j)| \leq \varepsilon/3$. We obtain $|\varphi(x) - \varphi(x_j)| \leq \varepsilon$ for all $j \geq j_0$. This shows that φ is strongly continuous on $(M)_1$ and hence $\varphi \in M_*$. □

Theorem 3.11. *Let M be any von Neumann algebra. The map $\Phi : M \to (M_*)^*$ defined by $\Phi(x)(\varphi) = \varphi(x)$ is an onto isometric linear map. Moreover, under the identification $M = (M_*)^*$, the ultraweak topology on M and the weak* topology on $(M_*)^*$ coincide.*

Proof. Assume $M \subset \mathbf{B}(H)$. For all $x \in M$, we have

$$\|x\|_\infty = \sup\{|\langle x\xi, \eta\rangle| : \xi, \eta \in H, \|\xi\| \leq 1, \|\eta\| \leq 1\}.$$

Put $\omega_{\xi,\eta} = \langle \cdot\, \xi, \eta\rangle$. Since $\omega_{\xi,\eta}|_M \in (M_*)_1$ for all $\xi, \eta \in H$ such that $\|\xi\| \leq 1$, $\|\eta\| \leq 1$, it follows that $\|x\|_\infty = \sup\{|\varphi(x)| : \varphi \in (M_*)_1\}$. Therefore Φ is an isometric embedding. It remains to show that Φ is onto.

Let $L \in (M_*)^*$. Define the bounded sesquilinear form b on $H \times H$ by $b(\xi, \eta) = L(\omega_{\xi,\eta}|_M)$. By Riesz Representation Theorem for sesquilinear forms, let $T \in \mathbf{B}(H)$ be the unique bounded operator such that $b(\xi, \eta) = \langle T\xi, \eta\rangle$ for all $\xi, \eta \in H$. Let $S \in M'$ be a selfadjoint element. For all $x \in M$, we have $\omega_{S\xi,\eta}(x) = \langle xS\xi, \eta\rangle = \langle Sx\xi, \eta\rangle = \langle x\xi, S\eta\rangle = \omega_{\xi,S\eta}(x)$ so that $\omega_{S\xi,\eta} = \omega_{\xi,S\eta}$. We obtain

$$\langle TS\xi, \eta\rangle = b(S\xi, \eta) = L(\omega_{S\xi,\eta}|_M) = L(\omega_{\xi,S\eta}|_M) = b(\xi, S\eta) = \langle ST\xi, \eta\rangle.$$

Therefore $T \in M'' = M$ by the Bicommutant Theorem. We have

$$\omega_{\xi,\eta}(T) = \langle T\xi, \eta\rangle = b(\xi, \eta) = L(\omega_{\xi,\eta}|_M).$$

Since any $\varphi \in M_*$ can be written $\varphi = \sum_n \omega_{\xi_n,\eta_n}|_M$ for some $(\xi_n)_n, (\eta_n)_n \in \ell^2(\mathbf{N}, H)$ (see Proposition 2.8) and since L is continuous, we obtain $\varphi(T) = L(\varphi)$ for all $\varphi \in M_*$. Thus $L = \Phi(T)$ and Φ is onto. □

Definition 3.12. Let M and N be any von Neumann algebras. A positive linear map $\pi : M \to N$ is *normal* if for every uniformly bounded increasing net of selfadjoint elements $(x_i)_{i\in I}$ in M, we have

$$\pi\left(\sup_{i\in I} x_i\right) = \sup_{i\in I} \pi(x_i).$$

We have the following characterization of normal states.

Theorem 3.13. *Let M be a von Neumann algebra together with a state $\varphi \in M^*$. The following are equivalent.*

(1) *φ is normal.*
(2) *Whenever $(p_i)_{i\in I}$ is a family of pairwise orthogonal projections in M, we have*

$$\varphi\left(\sum_{i\in I} p_i\right) = \sum_{i\in I} \varphi(p_i).$$

(3) *φ is ultraweakly continuous.*

Proof. (1) ⇒ (2). Let $(p_i)_{i\in I}$ be a family of pairwise orthogonal projections in M. Consider the increasing net $x_J = \sum_{i\in J} p_i$, where $J \subset I$ is a finite subset. We have $\sup_J x_J = \sum_{i\in I} p_i$ and so

$$\varphi\left(\sum_{i\in I} p_i\right) = \varphi\left(\sup_J x_J\right) = \sup_J \varphi(x_J) = \sup_J \sum_{i\in J} \varphi(p_i) = \sum_{i\in I} \varphi(p_i).$$

(2) ⇒ (3). Fix $q \in M$ a nonzero projection and $\xi \in \operatorname{ran}(q)$ such that $\varphi(q) \leq 1 < \langle q\xi, \xi\rangle$. There exists a nonzero projection $p \leq q$ such that $\varphi(pxp) \leq \langle pxp\xi, \xi\rangle$ for all $x \in M$. Indeed, by Zorn's Lemma, let $(p_i)_{i\in I}$ be a maximal family of pairwise orthogonal projections in M such that $\varphi(p_i) \geq \langle p_i\xi, \xi\rangle$ for all $i \in I$. By assumption, we have

$$\varphi\left(\sum_{i\in I} p_i\right) = \sum_{i\in I} \varphi(p_i) \geq \sum_{i\in I} \langle p_i\xi, \xi\rangle = \left\langle \left(\sum_{i\in I} p_i\right)\xi, \xi \right\rangle.$$

Put $p = q - \sum_{i\in I} p_i$ and observe that $p \neq 0$. By maximality of the family $(p_i)_{i\in I}$, we have $\varphi(r) < \langle r\xi, \xi\rangle$ for every nonzero projection $r \leq p$. Therefore, using the Spectral Theorem and since φ is $\|\cdot\|_\infty$-continuous, we get $\varphi(pxp) \leq \langle pxp\xi, \xi\rangle$ for all $x \in M_+$. By Cauchy–Schwarz Inequality, we have for all $x \in (M)_1$,

$$|\varphi(xp)|^2 = |\varphi(1^* xp)|^2 \leq \varphi(px^*xp)\varphi(1) \leq \langle px^*xp\xi, \xi\rangle = \|xp\xi\|^2.$$

It follows that $\varphi(\cdot\, p)$ is strongly continuous on $(M)_1$.

By Zorn's Lemma, let $(p_i)_{i\in I}$ be a maximal family of pairwise orthogonal projections such that $\varphi(\cdot\, p_i)$ is strongly continuous on $(M)_1$ for all $i \in I$. By maximality of the family and the previous reasoning, we have $\sum_{i\in I} p_i = 1$. Therefore $\sum_{i\in I} \varphi(p_i) = \varphi(1) = 1$. Let $\varepsilon > 0$. There exists a finite subset $F \subset I$ such for all finite subsets $F \subset J \subset I$, we have $\varphi(p_J^\perp) = 1 - \varphi(p_J) \leq \varepsilon$, where $p_J = \sum_{i\in J} p_i$. Moreover the Cauchy–Schwarz Inequality yields $|\varphi(xp_J^\perp)|^2 \leq \varphi(p_J^\perp)\varphi(xx^*) \leq \varepsilon$ for all $x \in (M)_1$ and all $F \subset J \subset I$. We have $\|\varphi - \varphi(\cdot\, p_J)\| \leq \sqrt{\varepsilon}$ for all $F \subset J \subset I$. Since the net $(\varphi(\cdot\, p_J))_J$ converges to φ in M^* and since $\varphi(\cdot\, p_J) \in M_*$ for all finite subsets $J \subset I$, we have $\varphi \in M_*$. (3) ⇒ (1) is trivial. □

Lemma 3.14. *Let $M \subset \mathbf{B}(H)$ be any von Neumann algebra. Any $\varphi \in M_*$ is a linear combination of four elements in $(M_*)_+$.*

Proof. By Proposition 2.8, there exist $(\xi_n)_n, (\eta_n)_n \in \ell^2(\mathbf{N}, H)$ such that $\varphi(x) = \sum_n \langle x\xi_n, \eta_n\rangle$ for all $x \in M$. A simple calculation shows that we have

$$\forall x \in M, \quad \langle x\xi_n, \eta_n\rangle = \frac{1}{4}\sum_{k=0}^{3} \mathrm{i}^k \langle x(\xi_n + \mathrm{i}^k\eta_n), \xi_n + \mathrm{i}^k\eta_n\rangle.$$

Put $\varphi_k(x) = \sum_n \langle x(\xi_n + \mathrm{i}^k\eta_n), \xi_n + \mathrm{i}^k\eta_n\rangle$ for all $x \in M$. For every $0 \leq k \leq 3$, we have $\varphi_k \in (M_*)_+$ and

$$\varphi = \frac{1}{4}\sum_{k=0}^{3} \mathrm{i}^k \varphi_k.$$ □

Theorem 3.15. *Any $*$-isomorphism between von Neumann algebras is normal and ultraweakly continuous.*

Proof. Let $\pi : M \to N$ be a $*$-isomorphism. Let (x_i) be a uniformly bounded net of selfadjoint operators in M and write $x = \sup x_i$. We have $\pi(x_i) \leq \pi(x)$ so that $\sup \pi(x_i) \leq \pi(x)$. Write

$y = \sup \pi(x_i)$. We have $x_i = \pi^{-1}(\pi(x_i)) \leq \pi^{-1}(y)$ so that $x \leq \pi^{-1}(y)$. Thus $y = \pi(x)$ and π is normal.

For all $\varphi \in (N_*)_+$, $\varphi \circ \pi$ is normal and thus ultraweakly continuous by Theorem 3.13. By Lemma 3.14, we have $\varphi \circ \pi \in M_*$ for all $\varphi \in N_*$. Therefore, π is ultraweakly continuous. □

3.3. Kaplansky's Density Theorem.

Theorem 3.16. *Let $\mathcal{A} \subset \mathbf{B}(H)$ be a unital $*$-subalgebra. Denote by M the strong closure of $\mathcal{A}$. The following are true.*

- *The strong closure of $(\mathcal{A})_1$ is $(M)_1$.*
- *The strong closure of $(\Re(\mathcal{A}))_1$ is $(\Re(M))_1$.*

Proof. We may assume that $\mathcal{A}$ is a unital C*-algebra. First assume that $x \in (\Re(M))_1$. Let $\mathcal{U}(x, \varepsilon, \xi_i)$ be a strong neighbourhood of x with $\varepsilon > 0$ and $\xi_1, \dots, \xi_n \in H$. Consider the continuous function $f(t) = 2t/(1+t^2)$ and observe that f is a homeomorphism from $[-1,1]$ onto itself. By continuous functional calculus, let $X \in (\Re(M))_1$ such that $x = f(X)$. By strong density of $\Re(\mathcal{A})$ in $\Re(M)$, we can find $Y \in \Re(\mathcal{A})$ such that

$$\|(Y-X)x\xi_i\| < \varepsilon \quad \text{and} \quad \left\|(Y-X)\frac{1}{1+X^2}\xi_i\right\| < \varepsilon/4 \quad \text{for all } 1 \leq i \leq n.$$

Define $y = f(Y)$ and observe that $y \in (\Re(\mathcal{A}))_1$. We have

$$\begin{aligned} y - x &= \frac{2Y}{1+Y^2} - \frac{2X}{1+X^2} \\ &= 2\frac{1}{1+Y^2}(Y(1+X^2) - (1+Y^2)X)\frac{1}{1+X^2} \\ &= 2\left(\frac{1}{1+Y^2}(Y-X)\frac{1}{1+X^2} + \frac{Y}{1+Y^2}(X-Y)\frac{X}{1+X^2}\right) \\ &= 2\frac{1}{1+Y^2}(Y-X)\frac{1}{1+X^2} + \frac{1}{2}y(X-Y)x. \end{aligned}$$

It follows that $\|(y-x)\xi_i\| < \varepsilon$ and so $y \in \mathcal{U}(x, \varepsilon, \xi_i)$.

Assume now that $x \in (M)_1$. Consider

$$a = \begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix} \in (\Re(\mathbf{M}_2(M)))_1.$$

The previous proof shows that there exists a net $b_i \in (\Re(\mathbf{M}_2(\mathcal{A})))_1$ of the form

$$b_i = \begin{pmatrix} \alpha_i & \beta_i \\ \beta_i^* & \gamma_i \end{pmatrix}$$

which converges strongly to a. Since $\|b_i\| \leq 1$, we have $\|\beta_i\| \leq 1$. Finally, we obtain that $\beta_i \to x$ strongly as $i \to \infty$. □

Corollary 3.17. *Let M be any von Neumann algebra, K any complex Hilbert space and $\pi : M \to \mathbf{B}(K)$ any unital ultraweakly continuous $*$-homomorphism. Then $\pi(M) \subset \mathbf{B}(K)$ is a von Neumann algebra.*

Proof. Observe that $\ker(\pi) \subset M$ is an ultraweakly closed two-sided $*$-ideal and hence of the form $\ker(\pi) = Mz$ where $z \in \mathcal{Z}(M)$ is a central projection. Up to restrincting $\pi|_{Mz} : Mz \to$

$\mathbf{B}(K)$, we may assume without loss of generality that $\pi : M \to \mathbf{B}(K)$ is a unital ultraweakly continuous $*$-isomorphism.

Denote by $\mathcal{M}$ the SOT closure of $\pi(M)$ in $\mathbf{B}(K)$. Let $T \in \mathcal{M}$. By Kaplansky Density Theorem, there exists a net $T_i \in \pi(M)$ such that $\|T_i\|_\infty \leq \|T\|_\infty$ for every $i \in I$ and $T_i \to T$ with respect to SOT as $i \to \infty$. Write $S_i = \pi^{-1}(T_i) \in M$ and observe that $\|S_i\|_\infty = \|T_i\|_\infty \leq \|T\|_\infty$ for every $i \in I$. Since $(M)_1$ is weakly compact and up to passing to a subnet, we may assume that there exists $S \in M$ such that $S_i \to S$ with respect to WOT as $i \to \infty$. Since π is ultraweakly continuous, we have that $T_i = \pi(S_i) \to \pi(S)$ with respect to WOT as $i \to \infty$. Since $T_i \to T$ with respect to SOT as $i \to \infty$, we have that $T_i \to T$ with respect to WOT as $i \to \infty$. By uniqueness of the limit, we obtain $T = \pi(S)$ and hence $\pi(M)$ is closed with respect to SOT. ☐

3.4. **Tracial von Neumann algebras.** A von Neumann algebra M is said to be *tracial* if it is endowed with a faithful normal state τ which satisfies the *trace* relation:

$$\tau(xy) = \tau(yx), \forall x, y \in M.$$

Such a tracial state will be refered to as a *trace*. We will say that M is a II_1 factor if M is an infinite dimensional tracial von Neumann algebra and a factor.

Let (M, τ) be a tracial von Neumann algebra. We endow M with the following inner product

$$\langle x, y\rangle_\tau = \tau(y^* x), \forall x, y \in M.$$

Denote by $(\pi_\tau, \mathrm{L}^2(M), \xi_\tau)$ the GNS representation of M with respect to τ. To simplify the notation, we identify $\pi_\tau(x)$ with $x \in M$ and regard $M \subset \mathbf{B}(\mathrm{L}^2(M))$. Define $J : M\xi_\tau \to \mathrm{L}^2(M) : x\xi_\tau \mapsto x^*\xi_\tau$. For all $x, y \in M$, we have

$$\langle Jx\xi_\tau, Jy\xi_\tau\rangle = \langle x^*\xi_\tau, y^*\xi_\tau\rangle = \tau(yx^*) = \tau(x^*y) = \langle y\xi_\tau, x\xi_\tau\rangle.$$

Thus $J : \mathrm{L}^2(M) \to \mathrm{L}^2(M)$ is a conjugate linear unitary such that $J^2 = 1$.

Theorem 3.18. *We have $JMJ = M'$.*

Proof. We first prove $JMJ \subset M'$. Let $x, y, a \in M$. We have

$$JxJy\, a\xi_\tau = Jxa^*y^*\xi_\tau = yax^*\xi_\tau = yJxa^*\xi_\tau = yJxJ\, a\xi_\tau$$

so that $JxJy = yJxJ$.

Claim 3.19. The faithful normal state $x \mapsto \langle x\xi_\tau, \xi_\tau\rangle$ is a trace on M'.

Let $x \in M'$. We first show that $Jx\xi_\tau = x^*\xi_\tau$. Indeed, for every $a \in M$, we have

$$\begin{aligned}\langle Jx\xi_\tau, a\xi_\tau\rangle &= \langle Ja\xi_\tau, x\xi_\tau\rangle = \langle x^*a^*\xi_\tau, \xi_\tau\rangle \\ &= \langle a^*x^*\xi_\tau, \xi_\tau\rangle = \langle x^*\xi_\tau, a\xi_\tau\rangle.\end{aligned}$$

Let now $x, y \in M'$. We have

$$\begin{aligned}\langle xy\xi_\tau, \xi_\tau\rangle &= \langle y\xi_\tau, x^*\xi_\tau\rangle = \langle y\xi_\tau, Jx\xi_\tau\rangle = \langle x\xi_\tau, Jy\xi_\tau\rangle \\ &= \langle x\xi_\tau, y^*\xi_\tau\rangle = \langle yx\xi_\tau, \xi_\tau\rangle.\end{aligned}$$

Put $\tau' : M' \to \mathbf{C} : x \mapsto \langle x\xi_\tau, \xi_\tau\rangle$. Define the canonical antiunitary K on $\mathrm{L}^2(M', \tau') = \overline{M'\xi_\tau} = \mathrm{L}^2(M)$ by $Kx\xi_\tau = x^*\xi_\tau$ for all $x \in M'$. The first part of the proof yields $KM'K \subset M'' = M$. Since K and J coincide on $M'\xi_\tau$, which is dense in $\mathrm{L}^2(M)$, it follows that $K = J$. Therefore, we have $JM'J \subset M$ and so $JMJ = M'$. ☐

Definition 3.20. Let $\mathcal{N} \subset \mathcal{M}$ be any inclusion of von Neumann algebras. A *conditional expectation* $\mathrm{E} : \mathcal{M} \to \mathcal{N}$ is a contractive unital $\mathcal{N}$-$\mathcal{N}$-bimodular linear map.

We next show that for inclusions of tracial von Neumann algebras $N \subset M$, there always exists a conditional expectation $\mathrm{E} : M \to N$.

Theorem 3.21. *Let $N \subset M$ be any inclusion of tracial von Neumann algebras and $\tau \in M_*$ a distinguished faithful normal trace. Then there exists a unique trace preserving conditional expectation $\mathrm{E}_N : M \to N$.*

Proof. We still denote by τ the faithful normal trace $\tau|_N \in N_*$. Regard $\mathrm{L}^2(N)$ as a closed subspace of $\mathrm{L}^2(M)$ via the identity mapping $\mathrm{L}^2(N) \to \mathrm{L}^2(M) : x\xi_\tau \mapsto x\xi_\tau$. For all $T \in M$, define a sesquilinear form $\kappa_T : \mathrm{L}^2(N) \times \mathrm{L}^2(N) \to \mathbf{C}$ by the formula

$$\kappa_T(x\xi_\tau, y\xi_\tau) = \tau(y^*Tx).$$

By Cauchy–Schwarz inequality, we have $|\kappa_T(x\xi_\tau, y\xi_\tau)| \leq \|T\|_\infty \|x\xi_\tau\| \|y\xi_\tau\|$ for all $x, y \in N$. By Riesz Representation Theorem, there exists $\mathrm{E}_N(T) \in \mathbf{B}(\mathrm{L}^2(N))$ such that $\kappa_T(x\xi_\tau, y\xi_\tau) = \langle \mathrm{E}_N(T)x\xi_\tau, y\xi_\tau\rangle$ for all $x, y \in N$. Observe that $\|\mathrm{E}_N(T)\|_\infty \leq \|T\|_\infty$. For all $x, y, a \in N$, we have

$$\begin{aligned}\langle \mathrm{E}_N(T)Ja^*J\,x\xi_\tau, y\xi_\tau\rangle &= \langle \mathrm{E}_N(T)xa\xi_\tau, y\xi_\tau\rangle \\ &= \tau(y^*Txa) \\ &= \tau((ya^*)^*Tx) \\ &= \langle \mathrm{E}_N(T)x\xi_\tau, ya^*\xi_\tau\rangle \\ &= \langle \mathrm{E}_N(T)x\xi_\tau, JaJy\xi_\tau\rangle \\ &= \langle Ja^*J\mathrm{E}_N(T)\,x\xi_\tau, y\xi_\tau\rangle.\end{aligned}$$

This implies that $\mathrm{E}(T) \in (JNJ)' = N$. It is routine to check that $\mathrm{E}_N : M \to N$ is a trace preserving conditional expectation.

We next show that there is a unique trace preserving conditional expectation $\mathrm{E} : M \to N$. Indeed, for all $T \in M$ and all $x, y \in N$, we have

$$\begin{aligned}\langle \mathrm{E}(T)x\xi_\tau, y\xi_\tau\rangle &= \tau(y^*\mathrm{E}(T)x) \\ &= \tau(\mathrm{E}(y^*Tx)) \\ &= \tau(y^*Tx) \\ &= \langle \mathrm{E}_N(T)x\xi_\tau, y\xi_\tau\rangle.\end{aligned}$$

This shows that $\mathrm{E}(T) = \mathrm{E}_N(T)$ for every $T \in M$ and hence $\mathrm{E} = \mathrm{E}_N$. □

3.5. **Type II_1 factors.** The next result is a WOT version of Dixmier averaging property.

Theorem 3.22 (WOT-Dixmier property)**.** *Let M be any type II_1 factor. Then for every $x \in M$, we have*

$$\overline{\mathrm{co}\,\{uxu^* : u \in \mathcal{U}(M)\}}^{\mathrm{WOT}} \cap \mathbf{C}1 \neq \emptyset.$$

In particular, there exists a unique normal tracial state on M.

Proof. Let $x \in M$ be any element. Denote by $\mathcal{K}_x = \overline{\mathrm{co}\,\{uxu^* : u \in \mathcal{U}(M)\}}^{\mathrm{WOT}}$ the weak closure of the convex hull of the uniformly bounded set $\{uxu^* : u \in \mathcal{U}(M)\}$. Observe that $\mathcal{K}_x \subset M$ is a uniformly bounded WOT-closed convex subset of M. We claim that $\mathcal{K}_x\xi_\tau \subset \mathrm{L}^2(M)$ is a weakly closed convex subset. Indeed, let $(x_i)_{i \in I}$ be a net in $\mathcal{K}_x$ and $\eta \in \mathrm{L}^2(M)$ such that $x_i \xi_\tau \to \eta$ weakly as $i \to \infty$. Since $(x_i)_{i \in I}$ is uniformly bounded, up to passing to a subnet, we may assume that there exists $y \in M$ such that $x_i \to y$ with respect to WOT as $i \to \infty$. Since $\mathcal{K}_x$ is WOT-closed, we have $y \in \mathcal{K}_x$. Since $x_i \to y$ with respect to WOT as $i \to \infty$, it follows that $x_i \xi_\tau \to y \xi_\tau$ weakly as $i \to \infty$. By uniqueness of the limit, we have $\eta = y \xi_\tau$. This shows that $\mathcal{K}_x \xi_\tau \subset \mathrm{L}^2(M)$ is a weakly closed convex subset. By Hahn–Banach Theorem, $\mathcal{K}_x \xi_\tau \subset \mathrm{L}^2(M)$ is a $\|\cdot\|$-closed convex subset.

Denote by $z \in \mathcal{K}_x$ the unique element such that $z\xi_\tau$ has minimal $\|\cdot\|_2$-norm. Observe that $uJuJ\,\mathcal{K}_x = \mathcal{K}_x$ and $\|uJuJ\,z\xi_\tau\| = \|z\xi_\tau\|$ for every $u \in \mathcal{U}(M)$. By uniqueness, it follows that for every $u \in \mathcal{U}(M)$, we have $uJuJ\,z\xi_\tau = z\xi_\tau$, that is, $uzu^* = z$ and hence $z \in \mathbf{C}1$. This shows that $z \in \mathcal{K}_x \cap \mathbf{C}1 \neq \emptyset$.

Let τ be any normal tracial state on M. By traciality and ultraweak continuity, we have that τ is constant on $\mathcal{K}_x$ and moreover $\mathcal{K}_x \cap \mathbf{C}1 = \{\tau(x)1\}$. This shows that τ is indeed unique. ☐

Theorem 3.23 (Equivalence of projections)**.** *Let M be any type II_1 factor. Denote by τ the unique (faithful) normal trace on M. Let $p, q \in M$ be any projections. The following assertions are equivalent:*

(1) $\tau(p) = \tau(q)$.
(2) *There exists $u \in \mathcal{U}(M)$ such that $upu^* = q$.*

Proof. Since (2) $\Rightarrow$ (1) is obvious, we only have to prove (1) $\Rightarrow$ (2). We may assume that $p \notin \{0,1\}$ so that $q \notin \{0,1\}$. We claim that there exists a nonzero partial isometry $v \in M$ such that $v^*v \leq p$ and $vv^* \leq q$. Indeed, since $p, q \neq 0$ and since M is a factor, there exists $x \in M$ such that $qxp \neq 0$. Write $qxp = v|qxp|$ for the polar decomposition of $qxp \in M$. Then $v \in M$ is a nonzero partial isometry such that $v^*v \leq p$ and $vv^* \leq q$.

Next, denote by J the directed set of all families $((p_i)_{i\in I}, (q_i)_{i \in I})$ such that $p_i \leq p$ and $q_i \leq q$ for all $i \in I$; the projections $(p_i)_{i \in I}$ (resp. $(q_i)_{i\in I}$) are pairwise orthogonal; for every $i \in I$, there exists a partial isometry $v_i \in M$ such that $v_i^* v_i = p_i$ and $v_i v_i^* = q_i$. The set J is clearly inductive. By Zorn's Lemma, let $((p_i)_{i\in I}, (q_i)_{i \in I}) \in J$ be a maximal element. Assume by contradiction that $\sum_{i \in I} p_i \neq p$. Since $\tau(p) = \tau(q)$ and since $\tau(p_i) = \tau(q_i)$ for every $i \in I$, we also have $\sum_{i\in I} q_i \neq q$. Applying the previous claim to $p - \sum_{i \in I} p_i$ and $q - \sum_{i\in I} q_i$, we obtain a nonzero partial isometry $v \in M$ such that $v^*v \leq p - \sum_{i\in I} p_i$ and $vv^* \leq q - \sum_{i \in I} q_i$. Then $(((p_i)_{i\in I}, v^*v), ((q_i)_{i\in I}, vv^*)) \in J$, which contradicts the maximality of $((p_i)_{i \in I}, (q_i)_{i\in I}) \in J$.

If we let $v = \sum_{i \in I} v_i$, then $v \in M$ is a partial isometry such that $v^*v = p$ and $vv^* = q$. Since $\tau(p^\perp) = \tau(q^\perp)$, the same reasoning as before shows that there exists a partial isometry $w \in M$ such that $w^*w = p^\perp$ and $ww^* = q$. Then $u = v + w \in \mathcal{U}(M)$ satisfies $upu^* = q$. ☐

3.6. **The hyperfinite type II_1 factor.** We start by proving a noncommutative version of the $\|\cdot\|_2$-convergence theorem for martingales.

Lemma 3.24 (Noncommutative martingales)**.** *Let (M, τ) be any tracial von Neumann algebra. Let $B_n \subset M$ be an increasing sequence of von Neumann subalgebras such that $\bigvee_{n \in \mathbf{N}} B_n = M$. For every $n \in \mathbf{N}$, denote by $\mathrm{E}_n : M \to B_n$ the unique trace preserving conditional expectation. The following assertions are true.*

- *For every $x \in M$, we have $\lim_n \|\mathrm{E}_n(x) - x\|_2 = 0$.*
- *Let $(x_n)_n$ be a uniformly bounded sequence in M such that $x_n \in B_n$ and $x_n = \mathrm{E}_{B_n}(x_{n+1})$ for all $n \in \mathbf{N}$. Then there exists $x \in M$ such that $x_n = \mathrm{E}_n(x)$ for all $n \in \mathbf{N}$ and $\lim_n \|x - x_n\|_2 = 0$.*

Proof. Denote by $e_n : \mathrm{L}^2(M) \to \mathrm{L}^2(B_n)$ the orthogonal projection corresponding to the conditional expectation $\mathrm{E}_n : M \to B_n$. Since $(B_n)_n$ is an increasing sequence of von Neumann subalgebras, we have that $(e_n)_n$ is an increasing sequence of projections in $\mathbf{B}(L^2(M))$. Since $\bigvee_{n\in\mathbf{N}} B_n = M$, we have $\bigvee_{n\in\mathbf{N}} e_n = 1$. It is easy to see that this implies that $\lim_n \|\mathrm{E}_n(x)-x\|_2 = 0$ for all $x \in M$.

Next, let $(x_n)_n$ be a uniformly bounded sequence in M such that $x_n \in B_n$ and $x_n = \mathrm{E}_{B_n}(x_{n+1})$ for all $n \in \mathbf{N}$. We have $\mathrm{E}_n(x_p) = x_n$ for all $p \geq n$. Let $x \in M$ be a weak limit point for the sequence $(x_n)_n$. In particular, we have $\mathrm{E}_n(x) = x_n$ for all $n \in \mathbf{N}$. By the first item, we moreover have $\lim_n \|x - x_n\|_2 = 0$. ☐

Let $\mathcal{A}_n := \mathbf{M}_{2^n}(\mathbf{C})$ and regard $\mathcal{A}_n \subset \mathcal{A}_{n+1}$ via the unital embedding

$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}.$$

This unital embedding preserves the normalized trace on matrices as well as the uniform norm. Let $\mathcal{A}_\infty = \bigcup_{n\in\mathbf{N}} \mathcal{A}_n$ and observe that $\mathcal{A}_\infty$ is a unital $*$-algebra endowed with a norm $\|\cdot\|$ which satisfies $\|a^*a\| = \|a\|^2$ for all $a \in \mathcal{A}_\infty$. Moreover, the normalized trace on matrices induces a faithful trace $\tau_\infty : \mathcal{A}_\infty \to \mathbf{C}$ which satisfies $|\tau_\infty(a)| \leq \|a\|$ and $\tau_\infty(a^*a) \geq 0$ for all $a \in \mathcal{A}_\infty$.

As we did for unital C*-algebras, we may perform the GNS construction for $(\mathcal{A}_\infty, \tau_\infty)$. We obtain a unital isometric $*$-representation

$$\pi_{\tau_\infty} : \mathcal{A}_\infty \to \mathbf{B}(\mathrm{L}^2(\mathcal{A}_\infty, \tau_\infty)) : a \mapsto (b\xi_{\tau_\infty} \mapsto ab\xi_{\tau_\infty}).$$

For simplicity, write $H = \mathrm{L}^2(\mathcal{A}_\infty, \tau_\infty)$.

Theorem 3.25. *We have that $R := \pi_{\tau_\infty}(\mathcal{A}_\infty)'' \subset \mathbf{B}(H)$ is a type II_1 factor and the vector state $\langle \cdot \xi_{\tau_\infty}, \xi_{\tau_\infty}\rangle$ defines a faithful normal trace on R.*

Proof. Let $\varphi = \langle \cdot \xi_{\tau_\infty}, \xi_{\tau_\infty}\rangle$ be the vector state defined on R. In order to show that φ is faithful, we have to show that ξ_{τ_∞} is separating for $\pi_{\tau_\infty}(\mathcal{A}_\infty)''$, that is, ξ_{τ_∞} is cyclic for $\pi_{\tau_\infty}(\mathcal{A}_\infty)'$. Define

$$J : \pi_{\tau_\infty}(\mathcal{A}_\infty) \to H : \pi_{\tau_\infty}(a)\xi_{\tau_\infty} \mapsto \pi_{\tau_\infty}(a^*)\xi_{\tau_\infty}.$$

As in the proof of Theorem 3.18, we check that J defines an antiunitary and that $J\pi_{\tau_\infty}(\mathcal{A}_\infty)''J \subset \pi_{\tau_\infty}(\mathcal{A}_\infty)'$. Since ξ_{τ_∞} is cyclic for $J\pi_{\tau_\infty}(\mathcal{A}_\infty)''J$, this implies that ξ_{τ_∞} is cyclic for $\pi_{\tau_\infty}(\mathcal{A}_\infty)'$ and hence ξ_{τ_∞} is separating for $\pi_{\tau_\infty}(\mathcal{A}_\infty)''$.

The state φ is clearly normal and since $\varphi(\pi_{\tau_\infty}(x)\pi_{\tau_\infty}(y)) = \varphi(\pi_{\tau_\infty}(y)\pi_{\tau_\infty}(x))$ for all $x, y \in \mathcal{A}_\infty$, we obtain that φ is a trace on R. We will simply denote it by τ from now on.

Observe that $Q_n := \pi_{\tau_\infty}(\mathcal{A}_n) = \pi_{\tau_\infty}(\mathcal{A}_n)''$ is an increasing sequence of finite dimensional von Neumann subalgebras of R such that $\bigvee_{n\in\mathbf{N}} Q_n = R$. Denote by $\mathrm{E}_n : R \to Q_n$ the unique trace preserving conditional expectation. Let $z \in \mathcal{Z}(R)$ and define $z_n = \mathrm{E}_n(z)$. We have $z_n \in \mathcal{Z}(Q_n)$, whence $z_n = \tau(z_n)1 = \tau(z)1$ for all $n \in \mathbf{N}$. Since $\lim_n \|z - z_n\|_2 = 0$, we have $z = \tau(z)1$. Therefore R is a type II_1 factor. ☐

The type II_1 factor R is called the *hyperfinite* type II_1 factor of Murray–von Neumann. In their seminal work [MvN43], Murray–von Neumann showed the uniqueness of the hyperfinite type II_1 factor.

Exercise 3.26. Let N be any type II_1 factor. Show that there exists a unital $*$-isomorphism $\pi : R \to N$.

4. Group von Neumann algebras and group measure space constructions

4.1. **Group von Neumann algebras.** Let Γ be a countable discrete group. The *left* regular representation $\lambda : \Gamma \to \mathcal{U}(\ell^2(\Gamma))$ is defined by $\lambda_s \delta_t = \delta_{st}$ for all $s, t \in \Gamma$.

Definition 4.1 (Group von Neumann algebra)**.** The *group von Neumann algebra* $\mathrm{L}(\Gamma)$ is defined as the weak closure of the linear span of $\{\lambda_s : s \in \Gamma\}$.

Likewise, we can define the *right* regular representation $\rho : \Gamma \to \mathcal{U}(\ell^2(\Gamma))$ by $\rho_s \delta_t = \delta_{ts^{-1}}$ for all $s, t \in \Gamma$. The *right* von Neumann algebra $\mathrm{R}(\Gamma)$ is defined as the weak closure of the linear span of $\{\rho_s : s \in \Gamma\}$. We obviously have $\mathrm{L}(\Gamma) \subset \mathrm{R}(\Gamma)'$.

Proposition 4.2. *The vector state $\tau : \mathrm{L}(\Gamma) \to \mathbf{C} : x \mapsto \langle x\delta_e, \delta_e\rangle$ is a faithful normal trace. Moreover $\mathrm{L}(\Gamma) = \mathrm{R}(\Gamma)'$.*

Proof. It is clear that τ is normal. We moreover have

$$\tau(\lambda_s\lambda_t) = \tau(\lambda_{st}) = \delta_{st,e} = \delta_{ts,e} = \tau(\lambda_{ts}) = \tau(\lambda_t\lambda_s).$$

It follows that τ is a trace on $\mathrm{L}(\Gamma)$. Assume now that $\tau(x^*x) = 0$, that is, $x\delta_e = 0$ for $x \in \mathrm{L}(\Gamma)$. For all $t \in \Gamma$, we have $x\delta_t = x\rho_{t^{-1}}\delta_e = \rho_{t^{-1}}x\delta_e = 0$. Therefore $x = 0$. Hence τ is faithful.

We can identify $\ell^2(\Gamma)$ with $\mathrm{L}^2(\mathrm{L}(\Gamma))$ via the unitary mapping $\delta_g \mapsto u_g$. Under this identification, we have $J\delta_t = \delta_{t^{-1}}$. An easy calculation shows that for all $s, t \in \Gamma$, we have

$$J\lambda_s J\, \delta_t = J\lambda_s\delta_{t^{-1}} = J\delta_{st^{-1}} = \delta_{ts^{-1}} = \rho_s\, \delta_t.$$

Therefore, $J\lambda_s J = \rho_s$ for all $s \in \Gamma$. It follows that $\mathrm{L}(\Gamma)' = J\mathrm{L}(\Gamma)J = \mathrm{R}(\Gamma)$ and thus $\mathrm{L}(\Gamma) = \mathrm{R}(\Gamma)'$. □

Let $x \in \mathrm{L}(\Gamma)$ and write $x\delta_e = \sum_{s\in\Gamma} x_s\delta_s \in \ell^2(\Gamma)$ with $x_s = \langle x\delta_e, \delta_s\rangle = \tau(x\lambda_s^*)$ for all $s \in \Gamma$. As we have seen, the family $(x_s)_{s\in\Gamma}$ completely determines $x \in \mathrm{L}(\Gamma)$. We shall denote by $x = \sum_{s\in\Gamma} x_s\lambda_s$ the *Fourier expansion* of $x \in \mathrm{L}(\Gamma)$.

The above sum $\sum_{s\in\Gamma} x_s\lambda_s$ **does not converge** in general for any of the topologies on $\mathbf{B}(\ell^2(\Gamma))$. However, the net of finite sums $(x_{\mathcal{F}})_{\mathcal{F}}$ defined by $x_{\mathcal{F}} = \sum_{s\in\mathcal{F}} x_s\lambda_s$ for $\mathcal{F} \subset \Gamma$ a finite subset does converge for the $\|\cdot\|_2$-norm. Indeed, since $(x_s) \in \ell^2(\Gamma)$, for any $\varepsilon > 0$, there exists $\mathcal{F}_0 \subset \Gamma$ finite subset such that $\sum_{s\in\Gamma\setminus\mathcal{F}_0} |x_s|^2 \leq \varepsilon^2$. Thus, for every finite subset $\mathcal{F} \subset \Gamma$ such that $\mathcal{F}_0 \subset \mathcal{F}$, we have $\|x - x_{\mathcal{F}}\|_2^2 = \sum_{s\in\Gamma\setminus\mathcal{F}} |x_s|^2 \leq \varepsilon^2$.

The notation $x = \sum_{s\in\Gamma} x_s\lambda_s$ behaves well with respect to taking the adjoint and multiplication.

Proposition 4.3. *Let $x = \sum_{s\in\Gamma} x_s\lambda_s$ (resp. $y = \sum_{t\in\Gamma} y_t\lambda_t$) be the Fourier expansion of $x \in \mathrm{L}(\Gamma)$ (resp. $y \in \mathrm{L}(\Gamma)$). Then we have*

- $x^* = \sum_{s\in\Gamma} \overline{x_{s^{-1}}}\lambda_s$.
- $xy = \sum_{t\in\Gamma} \left(\sum_{t\in\Gamma} x_s y_{s^{-1}t}\right)\lambda_t$*, with* $\sum_{s\in\Gamma} x_s y_{s^{-1}t} \in \mathbf{C}$ *for all* $t \in \Gamma$*, by Cauchy–Schwarz inequality.*

Proof. For the first item, observe that

$$(x^*)_s = \tau(x^*\lambda_s^*) = \overline{\tau(\lambda_s x)} = \overline{\tau(x\lambda_{s^{-1}}^*)} = \overline{x_{s^{-1}}}.$$

For the second item, observe that using Cauchy–Schwarz inequality, we have

$$(xy)_t = \tau(xy\lambda_t^*) = \sum_{s\in\Gamma} x_s\tau(\lambda_s y\lambda_t^*) = \sum_{s\in\Gamma} x_s\tau(y\lambda_{s^{-1}t}^*) = \sum_{s\in\Gamma} x_s y_{s^{-1}t}.$$

□

Thanks to the Fourier expansion, we can compute the center $\mathcal{Z}(\mathrm{L}(\Gamma))$ of the group von Neumann algebra. We say that Γ is icc (infinite conjugacy classes) if for every $s \in \Gamma \setminus \{e\}$, the conjugacy class $\{tst^{-1} : t \in \Gamma\}$ is infinite.

Proposition 4.4. *We have* $x = \sum_{s\in\Gamma} x_s\lambda_s \in \mathcal{Z}(\mathrm{L}(\Gamma))$ *if and only if* $x_{tst^{-1}} = x_s$ *for all* $s, t \in \Gamma$. *In particular,* $\mathrm{L}(\Gamma)$ *is a factor if and only if* Γ *is icc.*

Thus, $\mathrm{L}(\Gamma)$ *is a type* II_1 *factor whenever* Γ *is infinite and icc.*

Proof. We have

$$\begin{aligned} x = \sum_{s\in\Gamma} x_s\lambda_s \in \mathcal{Z}(\mathrm{L}(\Gamma)) &\Leftrightarrow \lambda_t^* x \lambda_t = x, \forall s \in \Gamma \\ &\Leftrightarrow x_{tst^{-1}} = x_s, \forall s, t \in \Gamma. \end{aligned}$$

If Γ is icc and $x \in \mathcal{Z}(\mathrm{L}(\Gamma))$, since $(x_{tst^{-1}})_t \in \ell^2(\Gamma)$, for all $s \in \Gamma$, it follows that $x_s = 0$ for all $s \in \Gamma \setminus \{e\}$. Hence $\mathcal{Z}(\mathrm{L}(\Gamma)) = \mathbf{C}1$.

If Γ is not icc, then $F = \{tst^{-1} : t \in \Gamma\}$ is finite for some $s \in \Gamma \setminus \{e\}$. Then $\sum_{h\in F} \lambda_h \in \mathcal{Z}(\mathrm{L}(\Gamma)) \setminus \mathbf{C}1$. □

Example 4.5. Here are a few examples of icc groups: the subgroup $S_\infty < S(\mathbf{N})$ of finitely supported permutations; the free groups $\mathbf{F}_n$ for $n \geq 2$; the lattices $\mathrm{PSL}(n, \mathbf{Z})$ for $n \geq 2$.

Hence Proposition 4.4 provides many examples of type II_1 factors arising from countable discrete groups.

Exercise 4.6. Let $T = [T_{st}]_{s,t\in\Gamma} \in \mathbf{B}(\ell^2(\Gamma))$, with $T_{st} = \langle T\delta_t, \delta_s\rangle$ for all $s, t \in \Gamma$. Show that $T \in \mathrm{L}(\Gamma)$ if and only if T is *constant down the diagonals*, that is, $T_{st} = T_{gh}$ whenever $st^{-1} = gh^{-1}$.

Example 4.7. Assume that Γ is a countable discrete abelian group. Then the Pontryagin dual $\widehat{\Gamma}$ is a compact second countable abelian group. Write $\mathcal{F} : \ell^2(\Gamma) \to \mathrm{L}^2(\widehat{\Gamma}, \mathrm{Haar})$ for the Fourier transform which is defined by $\mathcal{F}(\delta_s)(\chi) = \langle s, \chi\rangle$. Observe that $\mathcal{F}$ is a unitary operator. We have

$$\mathrm{L}^\infty(\widehat{\Gamma}) = \mathcal{F}\mathrm{L}(\Gamma)\mathcal{F}^*.$$

4.2. **Murray–von Neumann's group measure space construction.** Let $\Gamma \curvearrowright (X, \mu)$ be a probability measure preserving (pmp) action. Define the action $\sigma : \Gamma \curvearrowright \mathrm{L}^\infty(X)$ by $(\sigma_s(F))(x) = F(s^{-1}x)$ for all $F \in \mathrm{L}^\infty(X)$. This action extends to a unitary representation $\sigma : \Gamma \to \mathcal{U}(\mathrm{L}^2(X))$. Put $H = \mathrm{L}^2(X) \otimes \ell^2(\Gamma)$. Put $u_s = \sigma_s \otimes \lambda_s$ for all $s \in \Gamma$. Observe that by Fell's absorption principle, the representation $\Gamma \to \mathcal{U}(H) : s \mapsto u_s$ is unitarily conjugate to a multiple of the left regular representation. We will identify $F \in \mathrm{L}^\infty(X)$ with $F \otimes 1 \in \mathrm{L}^\infty(X) \otimes \mathbf{C}1$.

We have the following *covariance* relation:

$$u_s F u_s^* = \sigma_s(F), \forall F \in \mathrm{L}^\infty(X), \forall s \in \Gamma.$$

Definition 4.8 (Murray–von Neumann [MvN43])**.** The *group measure space construction* $\mathrm{L}^\infty(X) \rtimes \Gamma$ is defined as the weak closure of the linear span of $\{Fu_s : F \in \mathrm{L}^\infty(X), s \in \Gamma\}$.

Put $M = \mathrm{L}^\infty(X) \rtimes \Gamma$. Define the unital faithful $*$-representation $\pi : \mathrm{L}^\infty(X) \to \mathbf{B}(H)$ by $\pi(F)(\xi \otimes \delta_t) = \sigma_t(F)\xi \otimes \delta_t$. Denote by N the von Neumann algebra acting on H generated by $\pi(\mathrm{L}^\infty(X))$ and $(1 \otimes \rho)(\Gamma)$. It is straightforward to check that $M \subset N'$.

Proposition 4.9. *The vector state $\tau : M \to \mathbf{C}$ defined by $\tau(x) = \langle x(\mathbf{1}_X \otimes \delta_e), \mathbf{1}_X \otimes \delta_e\rangle$ is a faithful normal trace. Moreover we have $M = N'$.*

Proof. It is clear that τ is normal. We moreover have

$$\begin{aligned}\tau(Fu_s\, Gu_t) &= \tau(F\sigma_s(G)u_{st})\\ &= \delta_{st,e}\int_X F(x)G(s^{-1}x)\,\mathrm{d}\mu(x)\\ &= \delta_{st,e}\int_X F(sx)G(x)\,\mathrm{d}\mu(x)\\ &= \delta_{ts,e}\int_X G(x)F(t^{-1}x)\,\mathrm{d}\mu(x)\\ &= \tau(G\sigma_t(F)u_{ts})\\ &= \tau(Gu_t\, Fu_s).\end{aligned}$$

It follows that τ is a trace on M. Assume that $\tau(b^*b) = 0$, that is, $b(\mathbf{1}_X \otimes \delta_e) = 0$. For all $s \in \Gamma$ and all $F \in \mathrm{L}^\infty(X)$, we have

$$\begin{aligned}b\,(F \otimes \delta_t) &= b\,\pi(\sigma_{t^{-1}}(F))(1 \otimes \rho_{t^{-1}})(\mathbf{1}_X \otimes \delta_e)\\ &= \pi(\sigma_{t^{-1}}(F))(1 \otimes \rho_{t^{-1}})\,b(\mathbf{1}_X \otimes \delta_e) = 0.\end{aligned}$$

It follows that $b = 0$. Hence τ is faithful.

We will identify $\mathrm{L}^2(M)$ with $\mathrm{L}^2(X)\otimes\ell^2(\Gamma)$ via the unitary mapping $Fu_s\xi_\tau \mapsto F\otimes\delta_s$. Under this identification, the conjugation $J : \mathrm{L}^2(M) \to \mathrm{L}^2(M)$ is defined by $J(\xi \otimes \delta_s) = \sigma_{s^{-1}}(\xi^*) \otimes \delta_{s^{-1}}$. For all $F \in \mathrm{L}^\infty(X)$ and all $s \in \Gamma$, we have

$$\begin{aligned}J(\sigma_s \otimes \lambda_s)J &= 1 \otimes \rho_s\\ J(F \otimes 1)J &= \pi(F)^*.\end{aligned}$$

Therefore, we get $M = N'$. □

Observe that when the probability space $X = \{\bullet\}$ is a point, then the group von Neumann algebra and the group measure space construction coincide, that is, $\mathrm{L}^\infty(X) \rtimes \Gamma = \mathrm{L}(\Gamma)$.

Proposition 4.10 (Fourier expansion)**.** *Let $\Gamma \curvearrowright (X, \mu)$ be a pmp action. Let $A = \mathrm{L}^\infty(X)$ and $M = \mathrm{L}^\infty(X) \rtimes \Gamma$. Denote by $\mathrm{E}_A : M \to A$ the unique trace preserving conditional expectation. Every $a \in M$ has a unique* Fourier expansion *of the form $a = \sum_{s\in\Gamma} a_s u_s$ with $a_s = \mathrm{E}_A(au_s^*)$ for all $s \in \Gamma$. The convergence holds for the $\|\cdot\|_2$-norm. Moreover, we have the following:*

- $a^* = \sum_{s\in\Gamma} \sigma_{s^{-1}}(a_s^*)u_s$.
- $\|a\|_2^2 = \sum_{s\in\Gamma} \|a_s\|_2^2$.
- $ab = \sum_{t\in\Gamma} \left(\sum_{s\in\Gamma} a_s\sigma_s(b_{s^{-1}t})\right) u_t$.

Proof. Define the unitary mapping $U : \mathrm{L}^2(M) \to \mathrm{L}^2(X) \otimes \ell^2(\Gamma)$ by the formula $U(au_s\xi_\tau) = a \otimes \delta_s$. Then $U\xi_\tau = \mathbf{1}_X \otimes \delta_e$ is a cyclic separating vector for M represented on the Hilbert space $\mathrm{L}^2(X) \otimes \ell^2(\Gamma)$. We identify $\mathrm{L}^2(M)$ with $\mathrm{L}^2(X) \otimes \ell^2(\Gamma)$. Under this identification, e_A is the orthogonal projection $\mathrm{L}^2(X) \otimes \ell^2(\Gamma) \to \mathrm{L}^2(X) \otimes \mathbf{C}\delta_e$. Moreover, $u_se_Au_s^*$ is the orthogonal projection $\mathrm{L}^2(X) \otimes \ell^2(\Gamma) \to \mathrm{L}^2(X) \otimes \mathbf{C}\delta_s$ and thus $\sum_{s\in\Gamma} u_se_Au_s^* = 1$. Let $a \in M$. Regarding $a(\mathbf{1}_X \otimes \delta_e) \in \mathrm{L}^2(X) \otimes \ell^2(\Gamma)$, we know that there exists $a_s \in \mathrm{L}^2(X)$ such that

$$a(\mathbf{1}_X \otimes \delta_e) = \sum_{s\in\Gamma} a_s \otimes \delta_s \quad \text{and} \quad \|a\|_2^2 = \sum_{s\in\Gamma} \|a_s\|_2^2.$$

Then we have

$$\begin{aligned} a_s \otimes \delta_s &= u_s e_A u_s^* a(\mathbf{1}_X \otimes \delta_e) \\ &= u_s e_A u_s^* a e_A(\mathbf{1}_X \otimes \delta_e) \\ &= u_s \mathrm{E}_A(u_s^* a)(\mathbf{1}_X \otimes \delta_e) \\ &= \mathrm{E}_A(a u_s^*) \otimes \delta_s. \end{aligned}$$

It follows that $a_s = \mathrm{E}_A(au_s^*)$. Therefore, we have $a = \sum_{s\in\Gamma} \mathrm{E}_A(au_s^*)u_s$ and the convergence holds for the $\|\cdot\|_2$-norm. Moreover, $\|a\|_2^2 = \sum_{s\in\Gamma} \|\mathrm{E}_A(au_s^*)\|_2^2$. The rest of the proof is left to the reader. □

Like in the group case, the sum $a = \sum_{s\in\Gamma} a_s u_s$ **does not converge** in general for any of the operator topologies on $\mathbf{B}(\mathrm{L}^2(X) \otimes \ell^2(\Gamma))$.

Definition 4.11. Let $\Gamma \curvearrowright (X, \mu)$ be a pmp action.

- We say that the action is *(essentially) free* if $\mu(\{x \in X : sx = x\}) = 0$ for all $s \in \Gamma \setminus \{e\}$.
- We say that the action is *ergodic* if every Γ-invariant measurable subset $U \subset X$ has measure 0 or 1.

Lemma 4.12. *Let $\Gamma \curvearrowright (X, \mu)$ be a pmp action and denote by $\sigma : \Gamma \to \mathrm{L}^2(X)^0$ the corresponding Koopman representation where $\mathrm{L}^2(X)^0 = \mathrm{L}^2(X) \ominus \mathbf{C}\mathbf{1}_X$. The following are equivalent:*

1. *The action $\Gamma \curvearrowright (X, \mu)$ is ergodic.*
2. *The Koopman representation $\sigma \to \mathcal{U}(\mathrm{L}^2(X)^0)$ has no nonzero invariant vectors.*

Proof. (1) ⇒ (2) Let $\xi \in \mathrm{L}^2(X)^0$ such that $\sigma_s(\xi) = \xi$ for all $s \in \Gamma$. By considering the real part and the imaginary part of $\xi \in \mathrm{L}^2(X)^0$, we may further assume that $\xi \in \mathrm{L}^2(X)^0$ is real-valued. For every $t \in \mathbf{R}$, define $\mathcal{U}_t = \{x \in X : \xi(x) \geq t\}$. It follows that $\mathcal{U}_t$ is Γ-invariant for all $t \in \mathbf{R}$ and thus $\mu(\mathcal{U}_t) \in \{0, 1\}$ by ergodicity. Since the fonction $t \mapsto \mu(\mathcal{U}_t)$ is decreasing and since $\xi \in \mathrm{L}^2(X)$, there exists $t_0 \in \mathbf{R}$ such that $\mu(\mathcal{U}_t) = 1$ for all $t < t_0$ and $\mu(\mathcal{U}_t) = 0$ for all $t > t_0$. Therefore $\xi(x) = t_0$ for μ-almost every $x \in X$. Since $\xi \in \mathrm{L}^2(X)^0$, we get $t_0 = 0$ and so $\xi = 0$.

(2) ⇒ (1) Let $\mathcal{U} \subset X$ be a Γ-invariant measurable subset. Put $\xi = \mathbf{1}_\mathcal{U} - \mu(\mathcal{U})\mathbf{1}_X \in \mathrm{L}^2(X)^0$. Since $\sigma_s(\xi) = \xi$ for all $s \in \Gamma$, we get $\xi = 0$ and so $\mathbf{1}_\mathcal{U} = \mu(\mathcal{U})\mathbf{1}_X$. Hence $\mu(\mathcal{U}) \in \{0, 1\}$. □

Examples 4.13. Here are a few examples of pmp free ergodic actions $\Gamma \curvearrowright (X, \mu)$.

1. **Bernoulli actions.** Let Γ be an infinite group and (Y, η) a nontrivial probability space, that is, η is not a Dirac point mass. Put $(X, \mu) = (Y^\Gamma, \nu^{\otimes\Gamma})$. Consider the Bernoulli action $\Gamma \curvearrowright Y^\Gamma$ defined by
$$s \cdot (y_t)_{t\in\Gamma} = (y_{s^{-1}t})_{t\in\Gamma}.$$
Then the Bernoulli action is pmp free and mixing, so in particular ergodic.
2. **Profinite actions.** Let Γ be an infinite residually finite group together with a decreasing chain of finite index normal subgroups $\Gamma_n \lhd \Gamma$ such that $\Gamma_0 = \Gamma$ and $\cap_{n\in\mathbf{N}}\Gamma_n = \{e\}$. Then for all $n \geq 1$, the action $\Gamma \curvearrowright (\Gamma/\Gamma_n, \mu_n)$ is transitive and preserves the normalized counting measure μ_n. Consider the profinite action defined as the projective limit
$$\Gamma \curvearrowright (\mathbf{G}, \mu) = \varprojlim \Gamma \curvearrowright (\Gamma/\Gamma_n, \mu_n).$$
Then Γ sits as a dense subgroup of the compact group $\mathbf{G}$ which is the profinite completion of Γ with respect to the decreasing chain $(\Gamma_n)_{n\in\mathbf{N}}$. Observe that μ is the unique Haar probability measure on $\mathbf{G}$. The profinite action is pmp free and ergodic.

(3) **Actions on tori.** Let $n \geq 2$. Consider the action $\mathrm{SL}(n, \mathbf{Z}) \curvearrowright (\mathbf{T}^n, \lambda_n)$ where $\mathbf{T}^n = \mathbf{R}^n/\mathbf{Z}^n$ is the n-torus and λ_n is the unique Haar probability measure. This action is pmp free and ergodic.

We always assume that (X, μ) is a standard probability space. In particular, X is *countably separated* in the sense that there exists a sequence of Borel subsets $\mathcal{V}_n \subset X$ such that $\bigcup_n \mathcal{V}_n = X$, $\mu(\mathcal{V}_n) > 0$ for all $n \in \mathbf{N}$ and with the property that whenever $x, y \in X$ and $x \neq y$, there exists $n \in \mathbf{N}$ for which $x \in \mathcal{V}_n$ and $y \notin \mathcal{V}_n$.

Proposition 4.14. *Let $\Gamma \curvearrowright (X, \mu)$ be a pmp action. Put $A = \mathrm{L}^\infty(X)$ and $M = \mathrm{L}^\infty(X) \rtimes \Gamma$.*

(1) *The action is free if and only if $A \subset M$ is maximal abelian, that is, $A' \cap M = A$.*
(2) *Under the assumption that the action is free, the action is ergodic if and only if M is a factor.*

Proof. (1) Assume that the action is free. Let $b \in A' \cap M$ and write $b = \sum_{s \in \Gamma} b_s u_s$ for its Fourier expansion. Then for all $a \in A$ and all $s \in \Gamma$, we have $a b_s = \sigma_s(a) b_s$. Fix $s \in \Gamma \setminus \{e\}$ and put $\mathcal{U}_s = \{x \in X : b_s(x) \neq 0, sx \neq x\}$. We have $\mathbf{1}_{\mathcal{U}_s} a = \mathbf{1}_{\mathcal{U}_s} \sigma_s(a)$ for all $a \in A$.

By assumption, we have $\mathcal{U}_s = \mathcal{U}_s \cap \bigcup_n (\mathcal{V}_n \cap s(\mathcal{V}_n)^c)$. So, if $\mu(\mathcal{U}_s) > 0$, there exists $n \in \mathbf{N}$ such that $\mu(\mathcal{U}_s \cap \mathcal{V}_n \cap s(\mathcal{V}_n)^c) > 0$. With $a = \mathbf{1}_{\mathcal{V}_n}$, we get $\mathbf{1}_{\mathcal{U}_s \cap \mathcal{V}_n} = \mathbf{1}_{\mathcal{U}_s} \mathbf{1}_{\mathcal{V}_n} = \mathbf{1}_{\mathcal{U}_s} \sigma_s(\mathbf{1}_{\mathcal{V}_n}) = \mathbf{1}_{\mathcal{U}_s \cap s(\mathcal{V}_n)}$ and thus $\mathbf{1}_{\mathcal{U}_s \cap \mathcal{V}_n \cap s(\mathcal{V}_n)^c} = 0$, which is a contradiction. Therefore, $\mu(\mathcal{U}_s) = 0$. Since the action is moreover free, we get $b_s = 0$. This implies that $b \in A$.

Conversely, assume that $A' \cap M = A$. For all $s \in \Gamma \setminus \{e\}$, put $a_s = \mathbf{1}_{\{x \in X : sx = x\}}$. We have $a_s u_s \in A' \cap M = A$. Hence $a_s u_s = \mathrm{E}_A(a_s u_s) = 0$ and so $a_s = 0$. Therefore $\mu(\{x \in X : sx = x\}) = 0$.

(2) Under the assumption that the action is free, we have $\mathcal{Z}(M) = M' \cap M = M' \cap A = A^\Gamma$. Therefore, the action is ergodic if and only if $\mathcal{Z}(M) = \mathbf{C}1$. ☐

Let $A \subset M$ be any inclusion of von Neumann algebras. Denote by $\mathcal{N}_M(A) := \{u \in \mathcal{U}(M) : uAu^* = A\}$ the group of unitaries normalizing A inside M and by $\mathcal{N}_M(A)''$ the *normalizer* of A inside M. We say that $A \subset M$ is a *Cartan subalgebra* when the following three conditions are satisfied:

(1) A is maximal abelian, that is, $A = A' \cap M$;
(2) There exists a faithful normal conditional expectation $\mathrm{E}_A : M \to A$;
(3) $\mathcal{N}_M(A)'' = A$.

For every free pmp action $\Gamma \curvearrowright (X, \mu)$, $\mathrm{L}^\infty(X) \subset \mathrm{L}^\infty(X) \rtimes \Gamma$ is a Cartan subalgebra by Proposition 4.14.

5. Amenable von Neumann algebras

5.1. **Connes's theory of bimodules.** The discovery of the appropriate notion of representations for von Neumann algebras, as so-called *correspondences* or *bimodules*, is due to Connes. Whenever M is a von Neumann algebra, we denote by M^{op} its opposite von Neumann algebra.

Definition 5.1. Let M, N be tracial von Neumann algebras. A Hilbert space $\mathcal{H}$ is said to be an *M-N-bimodule* if it comes equipped with two commuting normal unital $*$-representations $\lambda : M \to \mathbf{B}(\mathcal{H})$ and $\rho : N^{\mathrm{op}} \to \mathbf{B}(\mathcal{H})$. We shall intuitively write

$$x \xi y = \lambda(x) \rho(y^{\mathrm{op}}) \xi, \ \forall \xi \in \mathcal{H}, \forall x \in M, \forall y \in N.$$

We will sometimes denote by $\pi_{\mathcal{H}} : M \otimes_{\text{alg}} N^{\text{op}} \to \mathbf{B}(\mathcal{H})$ the unital $*$-representation associated with the M-N-bimodule structure on $\mathcal{H}$.

Examples 5.2. Here are important examples of bimodules:

(1) The identity M-M-bimodule $\mathrm{L}^2(M)$ with $x\xi y = x\, Jy^* J\xi$.
(2) The coarse M-N-bimodule $\mathrm{L}^2(M) \otimes \mathrm{L}^2(N)$ with $x(\xi \otimes \eta)y = (x\xi) \otimes (\eta y)$.
(3) For any τ-preserving automorphism $\theta \in \mathrm{Aut}(M)$, we regard $\mathrm{L}^2_\theta(M) = \mathrm{L}^2(M)$ with the following M-M-bimodule structure: $x\xi y = x\xi\theta(y)$.

We will say that two M-N-bimodules ${}_M\mathcal{H}_N$ and ${}_M\mathcal{K}_N$ are *isomorphic* and write ${}_M\mathcal{H}_N \cong {}_M\mathcal{K}_N$ if there exists a unitary mapping $U : \mathcal{H} \to \mathcal{K}$ such that

$$U(x\xi y) = xU(\xi)y, \forall \xi \in \mathcal{H}, \forall x \in M, \forall y \in N.$$

Like for unitary group representations, we can define a notion of *weak containment* of Hilbert bimodules. Let M, N be any tracial von Neumann algebras and ${}_M\mathcal{H}_N, {}_M\mathcal{K}_N$ any bimodules. Consider the unital $*$-representations $\pi_{\mathcal{H}} : M \otimes_{\text{alg}} N^{\text{op}} \to \mathbf{B}(\mathcal{H})$ and $\pi_{\mathcal{K}} : M \otimes_{\text{alg}} N^{\text{op}} \to \mathbf{B}(\mathcal{K})$.

Definition 5.3 (Weak containment)**.** We say that $\mathcal{H}$ is *weakly contained* in $\mathcal{K}$ and write $\mathcal{H} \subset_{\text{weak}} \mathcal{K}$ if $\|\pi_{\mathcal{H}}(T)\| \leq \|\pi_{\mathcal{K}}(T)\|$ for all $T \in M \otimes_{\text{alg}} N^{\text{op}}$.

Let $\pi : \Gamma \to \mathcal{U}(K_\pi)$ be a unitary representation of a countable discrete group Γ. Put $M = \mathrm{L}(\Gamma)$ and denote by $(\lambda_s)_{s \in \Gamma}$ the canonical unitaries in M. Define on $\mathcal{H}(\pi) = K_\pi \otimes \ell^2(\Gamma)$ the following M-M-bimodule structure. For all $\xi \in K_\pi$ and all $s, t \in \Gamma$, define

$$\lambda_s\,(\xi \otimes \delta_t) = \pi_s(\xi) \otimes \delta_{st}$$
$$(\xi \otimes \delta_t)\,\lambda_s = \xi \otimes \delta_{ts}.$$

It is clear that the right multiplication extends to the whole von Neumann algebra M. Observe now that the unitary representations $\pi \otimes \lambda$ and $1_{K_\pi} \otimes \lambda$ are unitarily conjugate. Indeed, define $U : K_\pi \otimes \ell^2(\Gamma) \to K_\pi \otimes \ell^2(\Gamma)$ by

$$U(\xi \otimes \delta_t) = \pi_t(\xi) \otimes \delta_t.$$

It is routine to check that U is a unitary and $U(1_{K_\pi} \otimes \lambda_s)U^* = \pi_s \otimes \lambda_s$ for every $s \in \Gamma$. Therefore, the left multiplication extends to M. Denote by $1_\Gamma : \Gamma \to \mathcal{U}(\mathbf{C})$ the trivial representation.

Proposition 5.4 (Representations and Bimodules)**.** *The formulae above endow the Hilbert space $\mathcal{H}(\pi) = K_\pi \otimes \ell^2(\Gamma)$ with a structure of M-M-bimodule. Moreover, the following assertions hold true:*

(1) ${}_M\mathcal{H}(1_\Gamma)_M \cong {}_M\mathrm{L}^2(M)_M$ *and* ${}_M\mathcal{H}(\lambda_\Gamma)_M \cong {}_M(\mathrm{L}^2(M) \otimes \mathrm{L}^2(M))_M$.
(2) *For all unitary Γ-representations π_1 and π_2 such that $\pi_1 \subset_{\text{weak}} \pi_2$, we have*

$${}_M\mathcal{H}(\pi_1)_M \subset_{\text{weak}} {}_M\mathcal{H}(\pi_2)_M.$$

Proof. The proof is left as an exercise. □

5.2. **Powers–Størmer's inequality.** For an inclusion of von Neumann algebra $M \subset \mathcal{N}$, we say that a state $\varphi \in \mathcal{N}^*$ is M-central if $\varphi(xT) = \varphi(Tx)$ for all $x \in M$ and all $T \in \mathcal{N}$. We will be using the following notation: for all $x \in M$, put $\overline{x} = (x^{\mathrm{op}})^* \in M^{\mathrm{op}}$.

Regarding $M \otimes_{\mathrm{alg}} M^{\mathrm{op}} \subset \mathbf{B}(\mathrm{L}^2(M) \otimes \mathrm{L}^2(M))$, we will denote by $\|\cdot\|_{\min}$ the operator norm on $M \otimes_{\mathrm{alg}} M^{\mathrm{op}}$ induced by $\mathbf{B}(\mathrm{L}^2(M) \otimes \mathrm{L}^2(M))$. It is called the *minimal tensor norm*. We will also denote by $M \mathbin{\overline{\otimes}} M^{\mathrm{op}} := (M \otimes_{\mathrm{alg}} M^{\mathrm{op}})'' \subset \mathbf{B}(\mathrm{L}^2(M) \otimes \mathrm{L}^2(M))$.

Let H be any complex Hilbert space. For every $p \geq 1$, define the pth-*Schatten class* $\mathcal{S}_p(H)$ by

$$\mathcal{S}_p(H) = \{T \in \mathbf{B}(H) : \mathrm{Tr}(|T|^p) < \infty\}.$$

It is a Banach space with norm given by $\|T\|_p = \mathrm{Tr}(|T|^p)^{1/p}$. Observe that $\mathcal{S}_1(H)$ is the space of *trace-class* operators and $\mathcal{S}_2(H)$ is the Hilbert space of Hilbert–Schmidt operators. It is also denoted by $\mathrm{HS}(H)$.

Let (M, τ) be a tracial von Neumann algebra. The unitary mapping $U : \mathrm{HS}(\mathrm{L}^2(M)) \to \mathrm{L}^2(M) \otimes \mathrm{L}^2(M)$ defined by $U(\langle \cdot , \eta\rangle \xi) = \xi \otimes J\eta$ is an M-M-bimodule isomorphism.

We will be using the following technical results.

Lemma 5.5. *Let A be any unital C^*-algebra, $u \in (A)_1$ and $\omega \in A^*$ any state. Then we have*

$$\max\{\|\omega - \omega(u\,\cdot)\|, \|\omega - \omega(\cdot\, u^*)\|, \|\omega - \omega \circ \mathrm{Ad}(u)\|\} \leq 2\sqrt{2|1 - \omega(u)|}.$$

Proof. Let $(\pi_\omega, \mathcal{H}_\omega, \xi_\omega)$ be the GNS representation associated with the state ω on A. Then $\omega(a) = \langle \pi_\omega(a)\xi_\omega, \xi_\omega\rangle$ for all $a \in A$. We have

$$\|\omega - \omega(\cdot\, u^*)\| \leq \|\xi_\omega - \pi_\omega(u)^*\xi_\omega\| \leq \sqrt{2(1 - \Re(\omega(u)))} \leq \sqrt{2|1 - \omega(u)|}.$$

Likewise, we get $\|\omega - \omega(u\,\cdot)\| \leq \sqrt{2|1 - \omega(u)|}$. Moreover, we have

$$\|\omega - \omega \circ \mathrm{Ad}(u)\| \leq 2\|\xi_\omega - \pi_\omega(u)^*\xi_\omega\| \leq 2\sqrt{2|1 - \omega(u)|}. \qquad \square$$

The previous lemma implies in particular that when $\omega(u) = 1$, then

$$\omega = \omega(\cdot\, u^*) = \omega(u\,\cdot) = \omega \circ \mathrm{Ad}(u).$$

Lemma 5.6 (Powers–Størmer's Inequality)**.** *Let H be any Hilbert space and $S, T \in \mathcal{S}_2(H)_+$. Then we have*

$$\|S - T\|_2^2 \leq \|S^2 - T^2\|_1 \leq \|S - T\|_2 \|S + T\|_2.$$

Before starting the proof, we make the following observations:

- Whenever $A, B \in \mathbf{B}(H)$ have finite rank and if we write $AB = U|AB|$ for the polar decomposition, by the Cauchy–Schwarz Inequality, we have
$$\|AB\|_1 = \mathrm{Tr}(|AB|) = \mathrm{Tr}(U^*AB) \leq \|U^*A\|_2\|B\|_2 \leq \|A\|_2\|B\|_2.$$
- Whenever $A, B \in \mathbf{B}(H)_+$ and A or B has finite rank, we have $\mathrm{Tr}(AB) \geq 0$. Indeed, without loss of generality, we may assume that B has finite rank and we write $B = \sum_{i=1}^n \lambda_i\langle \cdot , \xi_i\rangle\, \xi_i$. Then $AB = \sum_{i=1}^n \lambda_i\langle \cdot , \xi_i\rangle\, A\xi_i$ and so $\mathrm{Tr}(AB) = \sum_{i=1}^n \lambda_i\langle A\xi_i, \xi_i\rangle \geq 0$.

Proof of Lemma 5.6. We reproduce the elegant proof given in [BO08, Proposition 6.2.4]. First observe that using the Spectral Theorem, we may assume that S, T have both finite rank and still satisfy $S, T \geq 0$.

The identity

$$S^2 - T^2 = \frac{1}{2}((S+T)(S-T) + (S-T)(S+T)) \tag{5.1}$$

together with the first observation give the right inequality.

Put $p = \mathbf{1}_{[0,+\infty)}(S-T)$. We have $(S-T)p \geq 0$ and $(T-S)p^\perp \geq 0$. Observe that we also have

$$\begin{aligned} \operatorname{Tr}((S+T)(S-T)p) &= \operatorname{Tr}((S+T)p(S-T)) \\ &= \operatorname{Tr}((S-T)(S+T)p) \end{aligned} \tag{5.2}$$

$$\begin{aligned} \operatorname{Tr}((T+S)(T-S)p^\perp) &= \operatorname{Tr}((T+S)p^\perp(T-S)) \\ &= \operatorname{Tr}((T-S)(T+S)p^\perp). \end{aligned} \tag{5.3}$$

Then we have

$$\begin{aligned} \|S-T\|_2^2 &= \operatorname{Tr}((S-T)^2) \\ &= \operatorname{Tr}((S-T)^2 p + (S-T)^2 p^\perp) \\ &= \operatorname{Tr}((S-T)(S-T)p + (T-S)(T-S)p^\perp) \\ &\leq \operatorname{Tr}((S+T)(S-T)p + (T+S)(T-S)p^\perp) \quad \text{(using the second obsevation)} \\ &= \operatorname{Tr}((S^2-T^2)p + (T^2-S^2)p^\perp) \quad \text{(using (5.1), (5.2) and (5.3))} \\ &\leq \operatorname{Tr}(|S^2-T^2|p + |T^2-S^2|p^\perp) \quad \text{(using the second observation)} \\ &= \operatorname{Tr}(|S^2-T^2|) = \|S^2-T^2\|_1. \end{aligned}$$

□

5.3. **Connes's fundamental theorem.** This section is devoted to proving Connes's characterization of *amenability* for tracial von Neumann algebras.

Definition 5.7. Let $M \subset \mathbf{B}(H)$ be any von Neumann algebra with separable predual. We say that

- M is *amenable* if there exists a conditional expectation $\Phi : \mathbf{B}(H) \to M$.
- M is *hyperfinite* if there exists an increasing sequence of unital finite dimensional $*$-subalgebras $Q_n \subset M$ such that $M = \bigvee_n Q_n$.

Theorem 5.8 (Connes [Co75]). *Let (M, τ) be a tracial von Neumann algebra with separable predual. The following are equivalent:*

(1) *There exists a conditional expectation $\Phi : \mathbf{B}(\mathrm{L}^2(M)) \to M$.*
(2) *There exists an M-central state φ on $\mathbf{B}(\mathrm{L}^2(M))$ such that $\varphi|_M = \tau$.*
(3) *There exists a net of unit vectors $\xi_n \in \mathrm{L}^2(M) \otimes \mathrm{L}^2(M)$ such that $\lim_n \|x\xi_n - \xi_n x\|_2 = 0$ and $\lim_n \langle x\xi_n, \xi_n\rangle = \tau(x)$ for all $x \in M$.*
(4) ${}_M\mathrm{L}^2(M)_M \subset_{\text{weak}} {}_M(\mathrm{L}^2(M) \otimes \mathrm{L}^2(M))_M$.
(5) *For all $a_1, \dots, a_k, b_1, \dots, b_k \in M$, we have*
$$\left|\tau\left(\sum_{i=1}^k a_i b_i\right)\right| \leq \left\|\sum_{i=1}^k a_i \otimes b_i^{\mathrm{op}}\right\|_{\min}.$$
(6) *M is hyperfinite.*

Whenever $M = \mathrm{L}(\Gamma)$ is the von Neumann algebra of a countable discrete group, conditions $(1-6)$ are equivalent to:

(7) Γ *is amenable.*

Proof. We show that (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) $\Leftrightarrow$ (7) and (6) $\Rightarrow$ (1). The proof of (1) $\Rightarrow$ (6) is beyond the scope of these notes.

(1) $\Rightarrow$ (2) Put $\varphi = \tau \circ \Phi$.

(2) $\Rightarrow$ (3) Let φ be an M-central state on $\mathbf{B}(\mathrm{L}^2(M))$. Since the set of normal states is $\sigma(\mathbf{B}(\mathrm{L}^2(M))^*, \mathbf{B}(\mathrm{L}^2(M)))$-dense in the set of states, we may choose a net of normal states $(\varphi_j)_{j \in J}$ on $\mathbf{B}(\mathrm{L}^2(M))$ such that $\lim_j \varphi_j(T) = \varphi(T)$ for all $T \in \mathbf{B}(\mathrm{L}^2(M))$. We get that $\varphi_j \circ \mathrm{Ad}(u) - \varphi_j \to 0$ with respect to the $\sigma(\mathbf{B}(\mathrm{L}^2(M))_*, \mathbf{B}(\mathrm{L}^2(M)))$-topology for all $u \in \mathcal{U}(M)$. Using Hahn–Banach Theorem and up to replacing the net $(\varphi_j)_{j \in J}$ by a net $(\varphi'_k)_{k \in K}$ where each φ'_k is equal to a finite convex combination of some of the φ_j's, we may assume that $\lim_j \|\varphi_j \circ \mathrm{Ad}(u) - \varphi_j\| = 0$ for all $u \in \mathcal{U}(M)$. For every $j \in J$, let $T_j \in \mathcal{S}_1(\mathrm{L}^2(M))_+$ be the unique trace-class operator such that $\varphi_j(S) = \mathrm{Tr}(T_j S)$ for all $S \in \mathbf{B}(\mathrm{L}^2(M))$. We get $\|T_j\|_1 = 1$ and $\lim_j \|uT_j u^* - T_j\|_1 = 0$ for all $u \in \mathcal{U}(M)$. Put $\xi_j = T_j^{1/2} \in \mathcal{S}_2(\mathrm{L}^2(M))$ and observe that $\|\xi_j\|_2 = 1$. Since ξ_j is a Hilbert–Schmidt operator, we may regard $\xi_j \in \mathrm{L}^2(M) \otimes \mathrm{L}^2(M)$. By the Powers–Størmer Inequality, we get $\lim_j \|u\xi_j u^* - \xi_j\|_2 = 0$ for all $u \in \mathcal{U}(M)$. Moreover, we have

$$\lim_j \langle x\xi_j, \xi_j\rangle = \lim_j \mathrm{Tr}(T_j x) = \lim_j \varphi_j(x) = \varphi(x) = \tau(x), \forall x \in M.$$

(3) $\Rightarrow$ (4) Let $a_1, \dots, a_k, b_1, \dots, b_k \in M$ and put $T = \sum_{i=1}^k a_i \otimes {b_i}^{\mathrm{op}}$. Let $c, d \in M$. Then

$$\begin{aligned}
\left|\langle \pi_{\mathrm{L}^2(M)}(T)c\xi_\tau, d\xi_\tau\rangle\right| &= \left|\tau\left(\sum_{i=1}^k d^* a_i c b_i\right)\right| \\
&= \lim_n \left|\left\langle \sum_{i=1}^k d^* a_i c b_i \,\xi_n, \xi_n\right\rangle\right| \\
&= \lim_n \left|\left\langle \sum_{i=1}^k a_i \,\xi_n c\, b_i, d\xi_n\right\rangle\right| \\
&\leq \|\pi_{\mathrm{L}^2(M)\otimes \mathrm{L}^2(M)}(T)\| \,\lim_n \|\xi_n c\| \,\lim_n \|d\xi_n\| \\
&= \|\pi_{\mathrm{L}^2(M)\otimes \mathrm{L}^2(M)}(T)\| \,\|c\xi_\tau\| \|d\xi_\tau\|.
\end{aligned}$$

This implies that $\|\pi_{\mathrm{L}^2(M)}(T)\| \leq \|\pi_{\mathrm{L}^2(M)\otimes \mathrm{L}^2(M)}(T)\|$.

(4) $\Rightarrow$ (5) Let $a_1, \dots, a_k, b_1, \dots, b_k \in M$ and put $T = \sum_{i=1}^k a_i \otimes b_i^{\mathrm{op}}$. Since $\mathrm{L}^2(M) \otimes \mathrm{L}^2(M)$ is a left $M \mathbin{\overline{\otimes}} M^{\mathrm{op}}$-module, we have

$$\|\pi_{\mathrm{L}^2(M)\otimes \mathrm{L}^2(M)}(T)\| = \left\|\sum_{i=1}^k a_i \otimes b_i^{\mathrm{op}}\right\|_{\min}.$$

Since by assumption, we have $\|\pi_{\mathrm{L}^2(M)}(T)\| \leq \|\pi_{\mathrm{L}^2(M)\otimes \mathrm{L}^2(M)}(T)\|$, we get

$$\left|\tau\left(\sum_{i=1}^k a_i b_i\right)\right| = \left|\left\langle \pi_{\mathrm{L}^2(M)}(T)\xi_\tau, \xi_\tau\right\rangle\right| \leq \|\pi_{\mathrm{L}^2(M)}(T)\| \leq \left\|\sum_{i=1}^k a_i \otimes b_i^{\mathrm{op}}\right\|_{\min}.$$

(5) $\Rightarrow$ (2) Denote by $\Omega : M \otimes_{\mathrm{alg}} M^{\mathrm{op}} \to \mathbf{C}$ the $\|\cdot\|_{\min}$-bounded functional such that $\Omega(a \otimes b^{\mathrm{op}}) = \tau(ab)$. By the Hahn–Banach Theorem and since $M \otimes_{\mathrm{alg}} M^{\mathrm{op}} \subset \mathbf{B}(\mathrm{L}^2(M) \otimes \mathrm{L}^2(M))$, we may extend the functional Ω to $\mathbf{B}(\mathrm{L}^2(M) \otimes \mathrm{L}^2(M))$ without increasing the norm of Ω. We still

denote this extension by Ω. Since $\|\Omega\| = 1 = \Omega(1)$, Ω is a state on $\mathbf{B}(\mathrm{L}^2(M) \otimes \mathrm{L}^2(M))$. Since $\Omega(u \otimes \overline{u}) = \tau(uu^*) = 1$ for all $u \in \mathcal{U}(M)$, we have

$$\Omega(S(u \otimes \overline{u})) = \Omega(S) = \Omega((u \otimes \overline{u})S)$$

for all $S \in \mathbf{B}(\mathrm{L}^2(M) \otimes \mathrm{L}^2(M))$ and all $u \in \mathcal{U}(M)$ (see Lemma 5.5).

Put $\varphi(T) = \Omega(T \otimes 1^{\mathrm{op}})$ for all $T \in \mathbf{B}(\mathrm{L}^2(M))$. Observe that $\varphi(x) = \Omega(x \otimes 1^{\mathrm{op}}) = \tau(x)$ for all $x \in M$. Moreover, for all $T \in \mathbf{B}(\mathrm{L}^2(M))$ and all $u \in \mathcal{U}(M)$, we have

$$\begin{aligned}\varphi(uT) &= \Omega(uT \otimes 1^{\mathrm{op}}) = \Omega((u \otimes \overline{u})(T \otimes u^{\mathrm{op}})) \\ &= \Omega((T \otimes u^{\mathrm{op}})(u \otimes \overline{u})) = \Omega(Tu \otimes 1^{\mathrm{op}}) \\ &= \varphi(Tu).\end{aligned}$$

(2) $\Rightarrow$ (1) For all $T \in \mathbf{B}(\mathrm{L}^2(M))$, define the sesquilinear form $\kappa_T : \mathrm{L}^2(M) \times \mathrm{L}^2(M) \to \mathbf{C}$ by the formula

$$\kappa_T(x\xi_\tau, y\xi_\tau) = \varphi(y^* T x).$$

By Cauchy–Schwarz inequality, we have $|\kappa_T(x\xi_\tau, y\xi_\tau)| \leq \|T\|_\infty \|x\|_2 \|y\|_2$ for all $x, y \in M$ and hence there exists $\Phi(T) \in \mathbf{B}(\mathrm{L}^2(M))$ such that $\kappa_T(x\xi_\tau, y\xi_\tau) = \langle \Phi(T)x\xi_\tau, y\xi_\tau\rangle$ for all $x, y \in M$. Observe that $\|\Phi(T)\|_\infty \leq \|T\|_\infty$. For all $x, y, a \in M$, we have

$$\begin{aligned}\langle \Phi(T)Ja^*J\, x\xi_\tau, y\xi_\tau\rangle &= \langle \Phi(T)xa\xi_\tau, y\xi_\tau\rangle \\ &= \varphi(y^*Txa) \\ &= \varphi((ya^*)^*Tx) \\ &= \langle \Phi(T)x\xi_\tau, ya^*\xi_\tau\rangle \\ &= \langle \Phi(T)x\xi_\tau, JaJy\xi_\tau\rangle \\ &= \langle Ja^*J\Phi(T)\, x\xi_\tau, y\xi_\tau\rangle.\end{aligned}$$

This implies that $\Phi(T) \in (JMJ)' = M$. It is routine to check that $\Phi : \mathbf{B}(\mathrm{L}^2(M)) \to M$ is a conditional expectation.

(6) $\Rightarrow$ (1) Assume that $M = \bigvee_n Q_n$ with $Q_n \subset M$ an increasing sequence of unital finite dimensional $*$-subalgebras. Denote by μ_n the unique Haar probability measure on the compact group $\mathcal{U}(Q_n)$. Choose a nonprincipal ultrafilter ω on $\mathbf{N}$. For all $T \in \mathbf{B}(\mathrm{L}^2(M))$, put

$$\mathrm{E}(T) = \lim_{n \to \omega} \int_{\mathcal{U}(Q_n)} uTu^* \, \mathrm{d}\mu_n(u).$$

Then $\Phi : \mathbf{B}(\mathrm{L}^2(M)) \to M$ defined by $\Phi(T) = J\mathrm{E}(T)J$ is a conditional expectation.

Put $M = \mathrm{L}(\Gamma)$ and denote by $\lambda_s \in M$ the canonical unitaries.

(1) $\Rightarrow$ (7) Let $\varphi \in \mathbf{B}(\ell^2(\Gamma))^*$ be an $\mathrm{L}(\Gamma)$-central state such that $\varphi|_{\mathrm{L}(\Gamma)} = \tau$. Define a state $m \in \ell^\infty(\Gamma)^*$ by $m = \varphi|_{\ell^\infty(\Gamma)}$. Then m is a left invariant mean and Γ is amenable.

(7) $\Rightarrow$ (1) Simply put $M = \mathrm{L}(\Gamma)$ and identify $\mathrm{L}^2(M) = \ell^2(\Gamma)$. Since Γ is amenable, there exists a left invariant mean $m : \ell^\infty(\Gamma) \to \mathbf{C}$. For every $T \in \mathbf{B}(\ell^2(\Gamma))$, define the bounded sesquilinear form $\kappa_T : \ell^2(\Gamma) \times \ell^2(\Gamma) \to \mathbf{C} : (\xi, \eta) \mapsto m(\gamma \mapsto \langle \rho_\gamma T \rho_\gamma^* \xi, \eta\rangle)$. By Riesz Representation Theorem, there exists $\Phi(T) \in \mathbf{B}(\ell^2(\Gamma))$ such that $\langle \Phi(T)\xi, \eta\rangle = m(\gamma \mapsto \langle \rho_\gamma T \rho_\gamma^* \xi, \eta\rangle)$ for all $\xi, \eta \in \ell^2(\Gamma)$. Observe that $\Phi : \mathbf{B}(\ell^2(\Gamma)) \to \mathbf{B}(\ell^2(\Gamma)) : T \mapsto \Phi(T)$ is a contractive unital $\mathrm{L}(\Gamma)$-$\mathrm{L}(\Gamma)$-bimodular

linear map. It remains to show that $\Phi(T) \in \mathrm{L}(\Gamma)$ for every $T \in \mathbf{B}(\ell^2(\Gamma))$. Indeed, for all $g \in \Gamma$ and all $\xi, \eta \in \ell^2(\Gamma)$, we have

$$\begin{aligned}\langle \rho_g \Phi(T) \rho_g^* \xi, \eta \rangle &= \langle \Phi(T) \rho_g^* \xi, \rho_g^* \eta \rangle \\ &= m(\gamma \mapsto \langle \rho_\gamma T \rho_\gamma^* \, \rho_g^* \xi, \rho_g^* \eta \rangle) \\ &= m(\gamma \mapsto \langle \rho_{g\gamma} T \rho_{g\gamma}^* \xi, \eta \rangle) \\ &= m(\lambda_{g^{-1}} \circ (\gamma \mapsto \langle \rho_\gamma T \rho_\gamma^* \xi, \eta \rangle)) \\ &= m(\gamma \mapsto \langle \rho_\gamma T \rho_\gamma^* \xi, \eta \rangle) \\ &= \langle \Phi(T) \xi, \eta \rangle.\end{aligned}$$

This implies that $\rho_g \Phi(T) \rho_g^* = \Phi(T)$ for every $g \in \Gamma$ and hence $\Phi(T) \in \rho(\Gamma)' \cap \mathbf{B}(\ell^2(\Gamma)) = \mathrm{L}(\Gamma)$. Therefore $\Phi : \mathbf{B}(\ell^2(\Gamma)) \to \mathrm{L}(\Gamma)$ is a conditional expectation. $\square$

We say that a tracial von Neumann algebra (M, τ) is *diffuse* if there exists a sequence of unitaries $u_n \in \mathcal{U}(M)$ such that $u_n \to 0$ with respect to WOT as $n \to \infty$. One can show that M is diffuse if and only if M has no nonzero minimal projection.

We record the following well-known fact.

Proposition 5.9. *Let $M \subset \mathbf{B}(H)$ be any diffuse tracial von Neumann algebra. Then for any M-central state $\varphi \in \mathbf{B}(H)^*$ we have $\varphi|_{\mathbf{K}(H)} = 0$.*

Proof. Fix a sequence of unitaries $u_n \in \mathcal{U}(M)$ such that $u_n \to 0$ with respect to WOT as $n \to \infty$. For any $\xi \in H$, denote by $e_\xi : H \to \mathbf{C}\xi$ the corresponding orthogonal projection. Since $\varphi \in \mathbf{B}(H)^*$ is M-central, we have $\varphi(e_{u_k \xi}) = \varphi(u_k e_\xi u_k^*) = \varphi(e_\xi)$ for every $k \in \mathbf{N}$ and every $\xi \in H$. Write $\|T\|_\varphi = \varphi(T^*T)^{1/2}$ for every $T \in \mathbf{B}(H)$.

Fix $\xi \in H$ and $N \geq 1$. By Cauchy–Schwarz inequality, we have

$$\varphi(e_\xi) = \frac{1}{N} \sum_{i=1}^N \varphi(e_{u_{k_i}\xi}) = \frac{1}{N} \varphi\left(\sum_{i=1}^N e_{u_{k_i}\xi}\right) \leq \frac{1}{N} \left\| \sum_{i=1}^N e_{u_{k_i}\xi} \right\|_\varphi.$$

We may choose $k_1, \dots, k_N \in \mathbf{N}$ such that $\|e_{u_{k_j}\xi} \, e_{u_{k_i}\xi}\|_\infty = |\langle u_{k_j}\xi, u_{k_i}\xi \rangle| \leq \frac{1}{N}$ for all $1 \leq i < j \leq N$. Then we also have

$$\begin{aligned}\left\| \sum_{i=1}^N e_{u_{k_i}\xi} \right\|_\varphi^2 &= \sum_{i=1}^N \varphi(e_{u_{k_i}\xi}) + \sum_{1 \leq i \neq j \leq N} \varphi(e_{u_{k_j}\xi} \, e_{u_{k_i}\xi}) \\ &\leq N + 2 \sum_{1 \leq i < j \leq N} \|e_{u_{k_j}\xi} \, e_{u_{k_i}\xi}\|_\infty \\ &\leq N + N(N-1)\frac{1}{N} = 2N - 1.\end{aligned}$$

Thus, we obtain

$$\varphi(e_\xi) \leq \frac{\sqrt{2N-1}}{N}.$$

Since this holds for every $N \geq 1$, it follows that $\varphi(e_\xi) = 0$. By Cauchy–Schwarz inequality, we also have $\varphi(S e_\xi) = 0$ for every $S \in \mathbf{B}(H)$. It follows that $\varphi(T) = 0$ for every rank one operator $T \in \mathbf{B}(H)$ and hence $\varphi|_{\mathbf{K}(H)} = 0$. $\square$

Proposition 5.9 shows that whenever (M, τ) is a *diffuse* amenable tracial von Neumann algebra, neither the conditional expectation $\Phi : \mathbf{B}(\mathrm{L}^2(M)) \to M$ nor the M-central state $\varphi \in \mathcal{B}(\mathrm{L}^2(M))^*$ such that $\varphi|_M = \tau$ are normal.

Exercise 5.10. Let $\Gamma \curvearrowright (X, \mu)$ be a pmp action of a countable discrete group on a standard probability space. Show that $\mathrm{L}^\infty(X) \rtimes \Gamma$ is amenable if and only if Γ is amenable.

Exercise 5.11. Let $A \subset M$ be any inclusion of tracial von Neumann algebras. Assume that A is amenable. Show that for every $u \in \mathcal{N}_M(A)$, the von Neumann subalgebra $\langle A, u \rangle \subset M$ is amenable.

References

[BO08] N.P. Brown, N. Ozawa, C^**-algebras and finite-dimensional approximations.* Graduate Studies in Mathematics, **88**. American Mathematical Society, Providence, RI, 2008.

[Co75] A. Connes, *Classification of injective factors. Cases* II_1, II_∞, III_λ, $\lambda \neq 1$. Ann. of Math. **74** (1976), 73–115.

[MvN43] F. Murray, J. von Neumann, *Rings of operators.* IV. Ann. of Math. **44** (1943), 716–808.

Laboratoire de Mathématiques d'Orsay, Université Paris-Sud, CNRS, Université Paris-Saclay, 91405 Orsay, FRANCE

E-mail address: cyril.houdayer@math.u-psud.fr