SPECTRAL GAP IN VON NEUMANN ALGEBRAS AND APPLICATIONS

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ABSTRACT. These are the lecture notes for the YMC*A summer school held at KU Leuven during August 13–17, 2018. In Lecture 1, we review the fullness property for arbitrary factors and give examples of full factors of type II_1 and of type III. In Lecture 2, we review the strong ergodicity property for nonsingular group actions and we prove the fullness property of group measure space factors arising from arbitrary strongly ergodic actions of bi-exact groups (e.g. free groups) due to Houdayer–Isono. In Lecture 3, we give a proof of Connes' spectral gap theorem for full factors of type II_1 due to Marrakchi and we prove Marrakchi's spectral gap theorem for full factors of type III. In Lecture 4, we review Popa's intertwining theory and we prove a unique McDuff decomposition theorem due to Houdayer–Marrakchi–Verraedt.

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1. Lecture 1: Introduction to full factors

We introduce the fullness property for type II_1 factors in terms of central nets and then for arbitrary factors in terms of centralizing nets. We provide examples of full factors of type II_1 and of type III.

Full factors of type II_1. We say that a von Neumann algebra M is *tracial* if M possesses a faithful normal tracial state τ. A type II_1 factor M is an infinite dimensional tracial von Neumann algebra with trivial center. A tracial von Neumann algebra (M, τ) is *amenable* if there exists a state $\varphi \in \mathbf{B}(\mathrm{L}^2(M, \tau))^*$ such that $\varphi|_M = \tau$ and $\varphi(xT) = \varphi(Tx)$ for every $x \in M$ and every $T \in \mathbf{B}(\mathrm{L}^2(M, \tau))$.

Let M be any type II_1 factor. Denote by τ its (unique) faithful normal tracial state. Write $\|x\|_2 = \sqrt{\tau(x^*x)}^{1/2}$ for every $x \in M$. Let $(x_i)_{i\in I} \in \ell^\infty(I, M)$ be any uniformly bounded net. We say that $(x_i)_{i\in I}$ is

- *central* if $\lim_i \|x_i y - y x_i\|_2 = 0$ for every $y \in M$.
- *trivial* if $\lim_i \|x_i - \tau(x_i)1\|_2 = 0$.

We first introduce the *fullness* property for factors of type II_1.

Definition 1.1. Let M be any type II_1 factor. We say that M is *full* if every central uniformly bounded net is trivial.

Let us point out that when M has separable predual (or equivalently M acts on a separable Hilbert space), M is full if and only if every central uniformly bounded sequence is trivial.

Denote by R the hyperfinite factor of type II_1 defined by

$$R = \left(\bigcup_{n\in\mathbf{N}} \mathbf{M}_{2^n}(\mathbf{C})\right)'' = \overline{\bigotimes}_{n\in\mathbf{N}}(\mathbf{M}_2(\mathbf{C}), \mathrm{tr}_2)$$

where tr_2 is the normalized trace on $\mathbf{M}_2(\mathbf{C})$. By Murray–Neumann's result [MvN43], R is the unique hyperfinite factor of type II_1. By Connes' fundamental result [Co75b], R is in fact the unique amenable factor of type II_1 (with separable predual). For every $n \in \mathbf{N}$, denote by $\pi_n : \mathbf{M}_2(\mathbf{C}) \to R$ the trace preserving embedding of $\mathbf{M}_2(\mathbf{C})$ into R corresponding to the nth position. Define

$$u_n = \pi_n\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) \in R.$$

It is straightforward to see that $(u_n)_{n\in\mathbf{N}}$ is a central uniformly bounded sequence in R. Moreover, for every $n \in \mathbf{N}$, we have $u_n \in \mathcal{U}(R)$, $\tau(u_n) = 0$ and so $\|u_n - \tau(u_n)1\|_2 = 1$. This shows that $(u_n)_{n\in\mathbf{N}}$ is not trivial and therefore R is not full. One can then think of the fullness property for type II_1 factors as a strengthening of nonamenability.

In order to give examples of full factors of type II_1, we look at von Neumann algebras associated with countable groups. Let Γ be any countable group and $\pi : \Gamma \to \mathcal{U}(H_\pi)$ any unitary representation. We say that π has *almost invariant vectors* if there exists a net of unit vectors $(\xi_i)_{i\in I}$ in H_π such that $\lim_i \|\pi_g(\xi_i) - \xi_i\| = 0$ for every $g \in \Gamma$. We say that π has *spectral gap* if there exist $\kappa > 0$ and $g_1, \dots, g_k \in \Gamma$ such that

$$\forall \xi \in H_\pi, \quad \|\xi\|^2 \leq \kappa \sum_{j=1}^{k} \|\pi_{g_j}(\xi) - \xi\|^2. \tag{1.1}$$

Observe that (1.1) is equivalent to saying that the spectrum of the positive selfadjoint bounded operator $T = \sum_{j=1}^k |\pi_{g_j} - 1|^2$ is contained in $[\frac{1}{\kappa}, +\infty)$. It is easy to check that π does not have almost invariant vectors if and only if π has spectral gap. The proof is left to the reader.

Denote by $\lambda : \Gamma \to \mathcal{U}(\ell^2(\Gamma))$ (resp. $\rho : \Gamma \to \mathcal{U}(\ell^2(\Gamma))$) the left (resp. right) regular representation. Recall that the left (resp. right) group von Neumann algebra $\mathrm{L}(\Gamma)$ (resp. $\mathrm{R}(\Gamma)$) is defined by $\mathrm{L}(\Gamma) = \{\lambda_g \mid g \in \Gamma\}'' \subset \mathbf{B}(\ell^2(\Gamma))$ (resp. $\mathrm{R}(\Gamma) = \{\rho_g \mid g \in \Gamma\}'' \subset \mathbf{B}(\ell^2(\Gamma))$). We have $\mathrm{L}(\Gamma) = \mathrm{R}(\Gamma)'$. A countable group Γ is ICC or has *infinite conjugacy classes* if for every $g \in \Gamma \setminus \{e\}$, the conjugacy class $\{sgs^{-1} \mid s \in \Gamma\}$ is infinite. It is well-known that whenever Γ is an ICC countably infinite group, the group von Neumann algebra $\mathrm{L}(\Gamma)$ is a type II_1 factor (with separable predual).

Recall that Γ is *amenable* if the left regular representation $\lambda : \Gamma \to \mathcal{U}(\ell^2(\Gamma))$ has almost invariant vectors. Then Γ is amenable if and only if $\mathrm{L}(\Gamma)$ is amenable. Denote by $\mathrm{Ad} : \Gamma \to \mathcal{U}(\ell^2(\Gamma)) : g \mapsto \lambda_g \rho_g$ the *conjugation* representation. Since the unit vector δ_e is always Ad-invariant, we rather consider the conjugation representation Ad^0 on the Ad-invariant subspace $\ell^2(\Gamma) \ominus \mathbf{C}\delta_e = \ell^2(\Gamma \setminus \{e\})$. The next definition is due to Effros [Ef73].

Definition 1.2. Let Γ be any countable group. We say that Γ is *inner amenable* if $\mathrm{Ad}^0 : \Gamma \to \mathcal{U}(\ell^2(\Gamma \setminus \{e\}))$ has almost invariant vectors.

Observe that a non-inner amenable countably infinite group is necessarily ICC. Indeed, if Γ is not ICC, then there exists $g \in \Gamma \setminus \{e\}$ such that $C_g = \{sgs^{-1} \mid s \in \Gamma\}$ is finite. Then the nonzero vector $\eta = \sum_{h \in C_g} \delta_h \in \ell^2(\Gamma \setminus \{e\})$ is Ad^0-invariant.

Example 1.3. Here are some well-known examples of non-inner amenable groups.

(i) Free product groups $\Gamma = \Gamma_1 * \Gamma_2$ where $|\Gamma_1| \geq 3$ and $|\Gamma_2| \geq 2$ (in particular free groups $\mathbf{F}_n$ where $n \geq 2$);
(ii) ICC property (T) groups;
(iii) ICC Gromov-word hyperbolic groups.

The following proposition shows in particular that free groups are not inner amenable.

Proposition 1.4. *For every $i \in \{1, 2\}$, let Γ_i be any countable group. Assume that $|\Gamma_1| \geq 3$ and $|\Gamma_2| \geq 2$. Then $\Gamma = \Gamma_1 * \Gamma_2$ is not inner amenable.*

Proof. Put $H = \ell^2(\Gamma \setminus \{e\})$. For every $i \in \{1, 2\}$, denote by $H_i \subset H$ the closure of the linear span of all δ_g's where g is a reduced word in Γ that begins with a letter in $\Gamma_i \setminus \{e\}$ and denote by $P_i : H \to H_i$ the corresponding orthogonal projection. Observe that $H = H_1 \oplus H_2$. Choose $a, b \in \Gamma_1 \setminus \{e\}$ such that $a \neq b$ and $c \in \Gamma_2 \setminus \{e\}$. For simplicity, write $\mathrm{Ad}^0_g(\xi) = g\xi g^{-1}$ for every $g \in \Gamma$ and every $\xi \in H$. Observe that $aH_2a^{-1} \subset H_1$, $bH_2b^{-1} \subset H_1$, $aH_2a^{-1} \perp bH_2b^{-1}$ and $cH_1c^{-1} \subset H_2$. Then, for every $\xi \in H$, we have

$$
\begin{aligned}
\|P_2(a^{-1}\xi a)\|^2 + \|P_2(b^{-1}\xi b)\|_2^2 &= \|P_{aH_2a^{-1}}(\xi)\|^2 + \|P_{bH_2b^{-1}}(\xi)\|_2^2 \\
&\leq \|P_1(\xi)\|_2^2 \\
\|P_1(c^{-1}\xi c)\|_2^2 &= \|P_{cH_1c^{-1}}(\xi)\|_2^2 \\
&\leq \|P_2(\xi)\|_2^2.
\end{aligned}
\tag{1.2}
$$

Let now $(\xi_n)_n$ be any $\|\cdot\|_2$-bounded Ad-invariant sequence in H. Then (1.2) implies that

$$
\begin{aligned}
\sqrt{2} \cdot \limsup_n \|P_2(\xi_n)\|_2 &\leq \limsup_n \|P_1(\xi_n)\|_2 \\
\limsup_n \|P_1(\xi_n)\|_2 &\leq \limsup_n \|P_2(\xi_n)\|_2.
\end{aligned}
$$

Thus, we have $\limsup_n \|P_1(\xi_n)\|_2 = \limsup_n \|P_2(\xi_n)\|_2 = 0$ and so $\lim_n \|\xi_n\|_2 = 0$. This shows that Γ is not inner amenable. ☐

Murray–von Neumann showed in [MvN43] that the free group factor $\mathrm{L}(\mathbf{F}_2)$ is full and deduced that $\mathrm{L}(\mathbf{F}_2)$ is not isomorphic to the hyperfinite type II_1 factor R. The following result due to Effros [Ef73, Theorem] provides plenty of examples of full factors of type II_1.

Theorem 1.5. *Let Γ be any non-inner amenable countably infinite group. Then $\mathrm{L}(\Gamma)$ is a full factor of type II_1.*

Proof. Since $\mathrm{Ad}^0 : \Gamma \to \mathcal{U}(\ell^2(\Gamma \setminus \{e\}))$ has spectral gap, there exist $\kappa > 0$ and $g_1, \dots, g_k \in \Gamma$ such that

$$\forall \xi \in \ell^2(\Gamma \setminus \{e\}), \quad \|\xi\|^2 \leq \kappa \sum_{j=1}^{k} \| \mathrm{Ad}^0_{g_j}(\xi) - \xi\|^2. \tag{1.3}$$

Put $M = \mathrm{L}(\Gamma)$. For every $j \in \{1, \dots, k\}$, put $u_j = \lambda_{g_j}$. For every $x \in M$ and every $g \in \Gamma$, we have $\mathrm{Ad}_g(x\delta_e) = \lambda_g \rho_g \, x\delta_e = \lambda_g x \rho_g \delta_e = \lambda_g x \lambda_g^* \delta_e$. Applying (1.3) to $\xi = (x - \tau(x)1)\delta_e \in \ell^2(\Gamma) \ominus \mathbf{C}\delta_e = \ell^2(\Gamma \setminus \{e\})$, we obtain

$$\forall x \in M, \quad \|x - \tau(x)1\|_2^2 \leq \kappa \sum_{j=1}^{k} \|u_j x - x u_j\|_2^2. \tag{1.4}$$

Then (1.4) clearly implies that every central uniformly bounded sequence in M is trivial. Thus, $\mathrm{L}(\Gamma) = M$ is full. □

The converse to Effros' theorem does not hold. Indeed, Vaes [Va09] constructed an example of an inner amenable ICC countably infinite group Λ such that $\mathrm{L}(\Lambda)$ is a full factor.

Full factors. Let M be any von Neumann algebra. We denote by M_* its predual, by $\mathcal{Z}(M)$ its center, by $\mathcal{U}(M)$ its group of unitaries, by M_h its subspace of selfadjoint elements and by $\mathrm{Ball}(M)$ its unit ball with respect to the uniform norm. We denote by $(M, \mathrm{L}^2(M), J, \mathrm{L}^2(M)_+)$ the *standard form* of M. More precisely, we have $M \subset \mathbf{B}(\mathrm{L}^2(M))$, $J : \mathrm{L}^2(M) \to \mathrm{L}^2(M)$ is a conjugate linear isometry such that $J^2 = 1$ and $\mathrm{L}^2(M)_+ \subset \mathrm{L}^2(M)$ is a closed convex cone that satisfies

$$\mathrm{L}^2(M)_+ = \left\{\zeta \in \mathrm{L}^2(M) \mid \langle \zeta, \xi \rangle \geq 0, \forall \xi \in \mathrm{L}^2(M)_+ \right\}.$$

Furthermore, we have $JMJ = M'$; $J\xi = \xi$ for every $\xi \in \mathrm{L}^2(M)_+$; $xJxJ\xi \in \mathrm{L}^2(M)_+$ for every $x \in M$ and every $\xi \in \mathrm{L}^2(M)_+$; $JzJ = z^*$ for every $z \in \mathcal{Z}(M)$. By [Ha73], the standard form of M always exists and is unique in an appropriate sense. The Hilbert space $\mathrm{L}^2(M)$ is naturally endowed with a structure of M-M-bimodule defined as follows:

$$\forall x, y \in M, \forall \eta \in \mathrm{L}^2(M), \quad x\eta y = xJy^*J\eta.$$

Observe that for every $b \in M_h$ and every $\zeta \in \mathrm{L}^2(M)$ such that $J\zeta = \zeta$, we have $\|\zeta b\| = \|Jb^*J\zeta\| = \|b\zeta\|$. To any element $\varphi \in (M_*)_+$ corresponds a unique element $\xi_\varphi \in \mathrm{L}^2(M)_+$ such that $\varphi = \langle \cdot \, \xi_\varphi, \xi_\varphi \rangle$. For every $\varphi \in (M_*)_+$ and every $x \in M$, we simply write $\|x\|_\varphi = \varphi(x^*x)^{1/2} = \|x\xi_\varphi\|$ and $\|x\|_\varphi^\sharp = \varphi(x^*x + xx^*)^{1/2} = (\|x\xi_\varphi\|^2 + \|\xi_\varphi x\|^2)^{1/2}$.

Example 1.6. Let M be any tracial von Neumann algebra with a distinguished faithful normal tracial state τ. Denote by $(\pi_\tau, \mathrm{L}^2(M, \tau), \xi_\tau)$ the GNS construction. Regard $M \subset \mathbf{B}(\mathrm{L}^2(M, \tau))$. Define $J : \mathrm{L}^2(M, \tau) \to \mathrm{L}^2(M, \tau) : x\xi_\tau \mapsto x^*\xi_\tau$. Then J extends to a well-defined conjugate linear isometry such that $J^2 = 1$. Moreover, one checks that $(M, \mathrm{L}^2(M, \tau), J, \mathrm{L}^2(M_+, \tau))$ is the standard form of M.

We say that M is *σ-finite* if M possesses a faithful normal state. For every faithful state $\varphi \in M_*$, the norm $\|\cdot\|_\varphi$ (resp. $\|\cdot\|_\varphi^\sharp$) induces the strong (resp. $*$-strong) operator topology on $\mathrm{Ball}(M)$. For every faithful state $\varphi \in M_*$, the *centralizer* of φ in M is defined by

$$\begin{aligned} M_\varphi &= \{x \in M \mid x\xi_\varphi = \xi_\varphi x\} \\ &= \{x \in M \mid \forall y \in M, \varphi(xy) = \varphi(yx)\}. \end{aligned}$$

It is straightforward to check that $M_\varphi \subset M$ is a von Neumann subalgebra. We say that M is *amenable* if there exists a conditional expectation $\Phi : \mathbf{B}(\mathrm{L}^2(M)) \to M$. Note that when M is a tracial von Neumann algebra, this definition coincides with the one we introduced in the previous subsection.

Let M be any factor. Recall that

- M **is of type** I if M has a minimal projection. In that case, we have $M \cong \mathbf{B}(\ell^2(I))$ for some nonempty index set I.
- M **is of type** II if M has no minimal projection and M possesses a faithful normal semifinite trace Tr. If $\mathrm{Tr}(1) < +\infty$, we say that M is of type II_1. If $\mathrm{Tr}(1) = +\infty$, we say that M is of type II_∞.
- M **is of type** III otherwise. When M is σ-finite and of type III, all nonzero projections in M are Murray-von Neumann equivalent to 1.

The classification of type III factors into subtypes was obtained by Connes in [Co72]. Since we will not use those classification results, we will not dwell further on that.

To define the fullness property for arbitrary factors, it is more appropriate to use the M-M-bimodule structure of $\mathrm{L}^2(M)$ rather than the M-M-bimodule structure of M. This is because the right multiplication on M does not extend to a representation by bounded operators on $\mathrm{L}^2(M)$. Let $(x_i)_{i\in I} \in \ell^\infty(I, M)$ be any uniformly bounded net. We say that $(x_i)_{i\in I}$ is

- *central* if $x_i y - y x_i \to 0$ $*$-strongly for every $y \in M$.
- *centralizing* if $\lim_i \|x_i\eta - \eta x_i\| = 0$ for every $\eta \in \mathrm{L}^2(M)$.
- *trivial* if there exists a bounded net $(\lambda_i)_{i\in I}$ in $\mathbf{C}$ such that $x_i - \lambda_i 1 \to 0$ $*$-strongly.

We now introduce the fullness property for arbitrary factors.

Definition 1.7. Let M be any factor. We say that M is *full* if every centralizing uniformly bounded net is trivial.

Let us point out that when M has separable predual, M is full if and only if every centralizing uniformly bounded sequence is trivial. One next checks that Definition 1.7 coincides with Definition 1.1 when M is a type II_1 factor. This is a consequence of the following useful lemma.

Lemma 1.8. *Let M be any σ-finite von Neumann algebra and $(x_i)_{i\in I}$ any uniformly bounded net in M. The following assertions are equivalent:*

(i) *The net $(x_i)_{i\in I}$ is centralizing.*
(ii) *The net $(x_i)_{i\in I}$ is central and for some (or any) faithful state $\varphi \in M_*$, we have $\lim_i \|x_i\xi_\varphi - \xi_\varphi x_i\| = 0$.*

Proof. (i) $\Rightarrow$ (ii) Fix any faithful state $\varphi \in M_*$. Since $(x_i)_{i\in I}$ is centralizing, for $\eta = \xi_\varphi$ we have $\lim_i \|x_i\xi_\varphi - \xi_\varphi x_i\| = 0$. For every $y \in M$, since $(x_i)_{i\in I}$ is centralizing, for $\eta = y\xi_\varphi$ we have

$$\limsup_i \|(x_i y - y x_i)\xi_\varphi\| \leq \limsup_i \|x_i\eta - \eta x_i\| + \limsup_i \|y(\xi_\varphi x_i - x_i\xi_\varphi)\| = 0.$$

Likewise, for $\eta = \xi_\varphi y$ we have

$$\limsup_i \|(x_i y - y x_i)^* \xi_\varphi\| = \limsup_i \|\xi_\varphi (x_i y - y x_i)\| \\ \leq \limsup_i \|(\xi_\varphi x_i - x_i \xi_\varphi) y\| + \limsup_i \|x_i \eta - \eta x_i\| = 0.$$

This implies that $(x_i)_{i\in I}$ is central.

(ii) $\Rightarrow$ (i) Since ξ_φ is M-cyclic and since $(x_i)_{i\in I}$ is uniformly bounded, it suffices to prove that $\lim_i \|x_i \eta - \eta x_i\| = 0$ for every $\eta \in \mathrm{L}^2(M)$ of the form $\eta = y\xi_\varphi$ for $y \in M$. For every $y \in M$, we have

$$\limsup_i \|x_i\, y\xi_\varphi - y\xi_\varphi\, x_i\| \leq \limsup_i \|(x_i y - y x_i)\xi_\varphi\| + \limsup_i \|y(x_i \xi_\varphi - \xi_\varphi x_i)\| = 0.$$

Thus, $(x_i)_{i\in I}$ is centralizing. □

When M is a type II_1 factor with faithful normal tracial state τ, since any element $x \in M$ commutes with $\xi_\tau \in \mathrm{L}^2(M)$, Lemma 1.8 implies that any central uniformly bounded net is centralizing. Therefore, Definition 1.7 coincides with Definition 1.1 for type II_1 factors.

For every $\lambda \in (0,1)$, denote by R_λ the Powers factor of type III_λ defined by

$$(R_\lambda, \varphi_\lambda) = \overline{\bigotimes}_{n\in\mathbf{N}} (\mathbf{M}_2(\mathbf{C}), \psi_\lambda) \quad \text{where} \quad \psi_\lambda = \mathrm{tr}_2\left(\cdot \begin{pmatrix} \frac{1}{1+\lambda} & 0 \\ 0 & \frac{\lambda}{1+\lambda} \end{pmatrix}\right).$$

For every $n \in \mathbf{N}$, denote by $\pi_n : \mathbf{M}_2(\mathbf{C}) \to R_\lambda$ the state preserving embedding of $\mathbf{M}_2(\mathbf{C})$ into R_λ corresponding to the nth position, meaning that $\varphi_\lambda \circ \pi_n = \psi_\lambda$. Define

$$u_n = \pi_n\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) \in R_\lambda.$$

Using Lemma 1.8, it is straightforward to see that $(u_n)_{n\in\mathbf{N}}$ is a centralizing uniformly bounded sequence in R_λ. Moreover, for every $n \in \mathbf{N}$, we have $u_n \in \mathcal{U}(R_\lambda)$, $\varphi_\lambda(u_n) = 0$ and so $\|u_n - \varphi_\lambda(u_n)1\|_{\varphi_\lambda} = 1$. This shows that $(u_n)_{n\in\mathbf{N}}$ is not trivial and therefore R_λ is not full. More generally, it follows from the work and Connes and Haagerup on the classification of amenable factors (see [Co75b, Co85, Ha85]) that any non-type I amenable factor with separable predual is never full. One can then think of the fullness property for arbitrary factors as a strengthening of nonamenability.

Example 1.9. Here are some examples of full factors (possibly of type III):

(i) For every nonamenable group Γ and every von Neumann algebra $B \neq \mathbf{C}1$ endowed with a faithful normal state ψ, the Bernoulli crossed product

$$\left(\overline{\bigotimes}_{g\in\Gamma} (B, \psi)\right) \rtimes \Gamma$$

is a full factor (see [Co74, VV14]). It is a type III factor if and only if ψ is not tracial.

(ii) For every orthogonal representation $U : \mathbf{R} \curvearrowright H_\mathbf{R}$ such that $\dim H_\mathbf{R} \geq 2$, Shlyakhtenko's free Araki–Woods factor $\Gamma(H_\mathbf{R}, U)''$ is a full factor (see [Sh96, Sh97, Va04]). It is a type III factor if and only if $U \neq \mathrm{id}$. When $U = \mathrm{id}$, we have $\Gamma(H_\mathbf{R}, \mathrm{id})'' \cong \mathrm{L}(\mathbf{F}_{\dim H_\mathbf{R}})$.

(iii) For every bi-exact group Γ (e.g. free group) and every strongly ergodic free nonsingular action $\Gamma \curvearrowright (X, \mu)$, the group measure space factor $\mathrm{L}(\Gamma \curvearrowright X)$ is full (see [HI15b]). It is a type III factor if and only if there is no σ-finite Γ-invariant measure that is equivalent to μ on X. We will prove this result in Lecture 2.

Finally, let us point out that there is a subtle difference between central nets and centralizing nets. For instance, one can show that in the Powers factor R_λ, there exist central uniformly bounded sequences that are not centralizing. We refer to [AH12, Example 5.1] and the references therein for further details.

2. Lecture 2: Full group measure space factors

We introduce the strong ergodicity property for nonsingular group actions and we provide examples of strongly ergodic actions. We prove Choda's result stating that strongly ergodic free probability measure preserving actions of non-inner amenable groups give rise to full factors of type II_1. We finally prove Houdayer–Isono's result stating that arbitrary strongly ergodic free nonsingular actions of bi-exact groups give rise to full factors.

Strongly ergodic actions. Let Γ be any countable group, (X, μ) any standard probability space and $\Gamma \curvearrowright X$ any measurable action. We say that the action $\Gamma \curvearrowright X$ is *nonsingular* and write $\Gamma \curvearrowright (X, \mu)$ if for every $g \in \Gamma$, the pushforward measure $g_*\mu$ has the same null measurable subsets as the measure μ. Moreover, we say that the nonsingular action $\Gamma \curvearrowright X$ is

- *probability measure preserving* (*pmp* for short) if for every $g \in \Gamma$, we have $g_*\mu = \mu$.
- *essentially free* (*free* for short) if for μ-almost every $x \in X$, we have $\mathrm{Stab}_\Gamma(x) = \{e\}$.
- *ergodic* if every Γ-invariant measurable subset $U \subset X$ (meaning that $\mu(U \triangle gU) = 0$ for every $g \in \Gamma$) is trivial (meaning that $\mu(U)(1 - \mu(U)) = 0$).
- *amenable* if there exists a Γ-equivariant conditional expectation $\Phi : \mathrm{L}^\infty(\Gamma \times X) \to \mathrm{L}^\infty(X)$ where we view $\mathrm{L}^\infty(X) \subset \mathrm{L}^\infty(\Gamma \times X)$ as a von Neumann subalgebra.

Let us point out that any infinite group Γ admits an ergodic free pmp action, namely the *Bernoulli action* $\Gamma \curvearrowright ([0, 1], \mathrm{Leb}^{\otimes \Gamma})$. For every amenable countable group Γ, every nonsingular action $\Gamma \curvearrowright (X, \mu)$ is amenable. Every nonamenable countable group admits at least one amenable ergodic nonsingular action, namely the *Poisson boundary action*.

Let Γ be any countable group, (X, μ) any standard probability space and $\Gamma \curvearrowright (X, \mu)$ any ergodic nonsingular action. Recall that

- $\Gamma \curvearrowright (X, \mu)$ **is of type** I if $\Gamma \curvearrowright (X, \mu)$ is essentially transitive. In that case, we have $(\Gamma \curvearrowright X) \cong (\Gamma \curvearrowright I)$ where I is a countable set and $\Gamma \curvearrowright I$ is a transitive action.
- $\Gamma \curvearrowright (X, \mu)$ **is of type** II if $\Gamma \curvearrowright (X, \mu)$ is not essentially transitive and there exists a σ-finite Γ-invariant measure ν on X that is equivalent to μ. If $\nu(X) < +\infty$, we say that $\Gamma \curvearrowright (X, \mu)$ is of type II_1. If $\nu(X) = +\infty$, we say that $\Gamma \curvearrowright (X, \mu)$ is of type II_∞.
- $\Gamma \curvearrowright (X, \mu)$ **is of type** III otherwise.

Let now $(U_n)_{n\in\mathbf{N}}$ be any sequence of measurable subsets of X. We say that $(U_n)_{n\in\mathbf{N}}$ is

- *invariant* if $\lim_n \mu(U_n \triangle gU_n) = 0$ for every $g \in \Gamma$.
- *trivial* if $\lim_n \mu(U_n)(1 - \mu(U_n)) = 0$.

The following definition due to Schmidt [Sc79] is central in this lecture.

Definition 2.1. Let Γ be any countable group, (X, μ) any standard probability space and $\Gamma \curvearrowright (X, \mu)$ any nonsingular action. We say that $\Gamma \curvearrowright (X, \mu)$ is *strongly ergodic* if every invariant sequence is trivial.

Any strongly ergodic nonsingular action is obviously ergodic. It is easy to show that the notion of strong ergodicity does not depend on the measure μ but only on the measure class of μ. We leave the details to the reader.

Let $\mathbf{Z} \curvearrowright^T (X, \mu)$ be any ergodic free pmp action. By Rokhlin's lemma, for every $n \in \mathbf{N}$, there exists a measurable subset $V_n \subset X$ such that $V_n, T(V_n), \dots, T^n(V_n)$ are pairwise disjoint and $\mu(X \setminus \bigsqcup_{j=0}^n T^j(V_n)) < 1/(n + 1)$. Put $U_n = \bigsqcup_{j=0}^{\lfloor n/2 \rfloor} T^j(V_n)$. Then the sequence $(U_n)_{n\in\mathbf{N}}$ is invariant and nontrivial since $\lim_n \mu(U_n) = 1/2$. Thus, the action $\mathbf{Z} \curvearrowright^T (X, \mu)$ is not strongly ergodic. More generally, it follows from Connes–Feldman–Weiss result [CFW81] that any amenable nonsingular action $\Gamma \curvearrowright (X, \mu)$ on a diffuse standard probability space is never strongly ergodic. Therefore, like the fullness property for non-type I factors, one can think of strong ergodicity as a strengthening of nonamenability.

Example 2.2. Here are some examples of strongly ergodic free nonsingular actions (possibly of type III).

(i) For every nonamenable countable group Γ, the Bernoulli action $\Gamma \curvearrowright ([0,1]^\Gamma, \mathrm{Leb}^{\otimes \Gamma})$ is a strongly ergodic free pmp action.
(ii) For every connected simple Lie group $\mathbf{G}$ and every countable dense subgroup $\Gamma < \mathbf{G}$ with *algebraic entries*, the translation action $\Gamma \curvearrowright \mathbf{G}$ is a strongly ergodic free measure preserving action (see [BG06, BG10, BdS14, BISG15]).
(iii) For every connected simple Lie group $\mathbf{G}$ with finite center, every lattice $\Gamma < \mathbf{G}$ and every nonamenable closed subgroup $\mathbf{H} < \mathbf{G}$, the homogeneous action $\Gamma \curvearrowright \mathbf{G}/\mathbf{H}$ is a strongly ergodic free nonsingular action (see [Oz16]).

Full group measure space factors. We introduce the group measure space construction due to Murray–von Neumann [MvN43]. Let Γ be any countable group, (X, μ) any standard probability space and $\Gamma \curvearrowright (X, \mu)$ any nonsingular action. Denote by $\sigma : \Gamma \curvearrowright \mathrm{L}^\infty(X)$ the action defined by $\sigma_g(f)(x) = f(g^{-1} \cdot x)$ for every $f \in \mathrm{L}^\infty(X)$. Put $H = \mathrm{L}^2(X, \mu) \otimes \ell^2(\Gamma)$. Define the unital $*$-representation $\pi_\sigma : \mathrm{L}^\infty(X) \to \mathbf{B}(H)$ by $\pi_\sigma(f)(\xi \otimes \delta_h) = \sigma_h(f) \otimes \delta_h$ for every $f \in \mathrm{L}^\infty(X)$, every $\xi \in \mathrm{L}^2(X, \mu)$ and every $h \in \Gamma$. For every $g \in \Gamma$, put $u_g = 1 \otimes \lambda_g$. Then we have the following *covariance relation*:

$$\forall g \in \Gamma, \forall f \in \mathrm{L}^\infty(X), \quad u_g \pi_\sigma(f) u_g^* = \pi_\sigma(\sigma_g(f)).$$

Definition 2.3. The *group measure space* von Neumann algebra associated with $\Gamma \curvearrowright (X, \mu)$ is defined by

$$\mathrm{L}(\Gamma \curvearrowright X) = \{\pi_\sigma(f), u_g \mid f \in \mathrm{L}^\infty(X), g \in \Gamma\}'' \subset \mathbf{B}(H).$$

For simplicity, we identify $\mathrm{L}^\infty(X)$ with its image $\pi_\sigma(\mathrm{L}^\infty(X))$ in $\mathbf{B}(H)$ and regard $A = \mathrm{L}^\infty(X) \subset \mathrm{L}(\Gamma \curvearrowright X) = M$. The mapping $\mathrm{E}_A : M \to A : f u_g \mapsto f \delta_{e,g}$ extends to a well-defined faithful normal conditional expectation. Write $\tau = \int_X \cdot \, \mathrm{d}\mu$ and $\varphi = \tau \circ \mathrm{E}_{\mathrm{L}^\infty(X)}$. Then $\varphi \in M_*$ is a faithful state and $A \subset M_\varphi$. We write $x = \sum_{g \in \Gamma} x^g u_g$ for the Fourier expansion of $x \in M$. Note that for all measurable subsets $U, V \subset X$, we have $\|\mathbf{1}_U - \mathbf{1}_V\|_\varphi^2 = \mu(U \triangle V)$. This shows that for every measurable subset $U \subset X$, U is invariant if and only if $\mathbf{1}_U \in \mathcal{Z}(M)$. Moreover, for every sequence of measurable subsets $(U_n)_{n \in \mathbf{N}}$ in X, $(U_n)_{n \in \mathbf{N}}$ is invariant if and only if $(\mathbf{1}_{U_n})_{n \in \mathbf{N}}$ is centralizing. If the nonsingular action $\Gamma \curvearrowright (X, \mu)$ is free, then $A \subset M$ is maximal abelian, meaning that $A' \cap M = A$. In that case, $\Gamma \curvearrowright (X, \mu)$ is ergodic if and only if M is a factor. The nonsingular action $\Gamma \curvearrowright (X, \mu)$ is amenable if and only if M is amenable.

Proposition 2.4. *Let Γ be any countable group, (X, μ) any standard probability space and $\Gamma \curvearrowright (X, \mu)$ any nonsingular action. Put $A = \mathrm{L}^\infty(X)$ and $M = \mathrm{L}(\Gamma \curvearrowright X)$. The following assertions are equivalent:*

(i) *$\Gamma \curvearrowright (X, \mu)$ is strongly ergodic.*
(ii) *For every uniformly bounded sequence $(a_n)_n \in \ell^\infty(\mathbf{N}, A)$, if $\lim_n \|a_n - \sigma_g(a_n)\|_2 = 0$ for every $g \in \Gamma$, then $\lim_n \|a_n - \tau(a_n)1\|_2 = 0$.*

Proof. (ii) $\Rightarrow$ (i) Let $(U_n)_{n \in \mathbf{N}}$ be any invariant sequence of measurable subsets of X. For every $n \in \mathbf{N}$, put $p_n = \mathbf{1}_{U_n} \in A$. Since $(U_n)_{n \in \mathbf{N}}$ is invariant, we have $\lim_n \|p_n - \sigma_g(p_n)\|_2 = 0$ for every $g \in \Gamma$. Then we have

$$\lim_n \mu(U_n)(1 - \mu(U_n)) = \lim_n \|p_n - \tau(p_n)1\|_2^2 = 0.$$

(i) $\Rightarrow$ (ii) We use ultraproduct techniques. Let $(a_n)_n \in \ell^\infty(\mathbf{N}, A)$ be any uniformly bounded sequence such that $\lim_n \|a_n - \sigma_g(a_n)\|_2 = 0$ for every $g \in \Gamma$. Let $\omega \in \beta(\mathbf{N}) \setminus \mathbf{N}$ be any nonprincipal ultrafilter. Consider the tracial ultraproduct von Neumann algebra (A^ω, τ^ω) and the ultraproduct action $\sigma^\omega : \Gamma \curvearrowright A^\omega$. We claim that $(A^\omega)^\Gamma = \mathbf{C}1$. Indeed, let $p \in (A^\omega)^\Gamma$ be any projection. By [Co75a, Proposition 1.1.3 (a)], we may write $p = (p_n)^\omega$ where $p_n \in A$ is a projection for every $n \in \mathbf{N}$. Write $p_n = \mathbf{1}_{U_n}$ for some measurable subset $U_n \subset X$. Since for every $g \in \Gamma$, we have

$$\lim_{n\to\omega} \mu(U_n \triangle gU_n) = \lim_{n\to\omega} \|\mathbf{1}_{U_n} - \mathbf{1}_{gU_n}\|_2^2 = \lim_{n\to\omega} \|p_n - \sigma_g(p_n)\|_2^2 = \|p - \sigma_g^\omega(p)\|_2^2 = 0,$$

it follows that $(U_n)_{n\in\mathbf{N}}$ is an ω-invariant sequence. By strong ergodicity, we have

$$\|p - \tau^\omega(p)1\|_2^2 = \lim_{n\to\omega} \|p_n - \tau(p_n)1\|_2^2 = \lim_{n\to\omega} \mu(U_n)(1 - \mu(U_n)) = 0.$$

Thus, we have $p \in \{0, 1\}$. This shows that $(A^\omega)^\Gamma = \mathbf{C}1$. Since $(a_n)^\omega \in (A^\omega)^\Gamma$, it follows that $\lim_{n\to\omega} \|a_n - \tau(a_n)1\|_2 = \|(a_n)^\omega - \tau^\omega((a_n)^\omega)1\|_2 = 0$. Since this holds true for every $\omega \in \beta(\mathbf{N}) \setminus \mathbf{N}$, it follows that $\lim_n \|a_n - \tau(a_n)1\|_2 = 0$. ☐

Proposition 2.4 implies that for every ergodic free nonsingular action $\Gamma \curvearrowright (X, \mu)$, if $\mathrm{L}(\Gamma \curvearrowright X)$ is full then $\Gamma \curvearrowright (X, \mu)$ is strongly ergodic. We point out that the converse need not be true as demonstrated by Connes–Jones [CJ81]. However, it is natural to seek for natural classes of groups and actions for which strong ergodicity of the action implies fullness of the corresponding group measure space construction.

We first review a well-known result due to Choda [Ch81] that gives a satisfactory answer to the above question in the probability measure preserving setting.

Theorem 2.5. *Let Γ be any non-inner amenable group, (X, μ) any standard probability space and $\Gamma \curvearrowright (X, \mu)$ any strongly ergodic free pmp action. Then $\mathrm{L}(\Gamma \curvearrowright X)$ is a full factor.*

Proof. Let $(x_n)_{n\in\mathbf{N}} \in \ell^\infty(\mathbf{N}, M)$ be any central uniformly bounded sequence in M. For every $n \in \mathbf{N}$, write $x_n = \sum_{h\in\Gamma} (x_n)^h u_h$ for the Fourier expansion of $x_n \in M$. Put $\xi_n = \sum_{h\in\Gamma} \|(x_n)^h\|_2 \, \delta_h \in \ell^2(\Gamma)$. For every $g \in \Gamma$, we have

$$\begin{aligned}
\| \mathrm{Ad}_g(\xi_n) - \xi_n\|^2 &= \sum_{h\in\Gamma} \left| \|(x_n)^{g^{-1}hg}\|_2 - \|(x_n)^h\|_2 \right|^2 \\
&= \sum_{h\in\Gamma} \left| \|g \cdot (x_n)^{g^{-1}hg}\|_2 - \|(x_n)^h\|_2 \right|^2 \\
&\leq \sum_{h\in\Gamma} \left\| g \cdot (x_n)^{g^{-1}hg} - (x_n)^h \right\|_2^2 \\
&= \|u_g x_n u_g^* - x_n\|_2^2 \to 0 \quad \text{as} \quad n \to \infty.
\end{aligned}$$

Since Γ is not inner amenable, it follows that $\lim_n \|\xi_n - \|(x_n)^e\|_2 \, \delta_e\| = 0$ and so $\lim_n \|x_n - \mathrm{E}_A(x_n)\|_2 = 0$. Since $(x_n)_n$ is central, it follows that

$$\lim_n \|\mathrm{E}_A(x_n) - \sigma_g(\mathrm{E}_A(x_n))\|_2 = \lim_n \|\mathrm{E}_A(x_n - u_g x_n u_g^*)\|_2 = 0$$

for every $g \in \Gamma$. Then Proposition 2.4 implies that $\lim_n \|\mathrm{E}_A(x_n) - \tau(x_n)1\|_2 = \lim_n \|\mathrm{E}_A(x_n) - \tau(\mathrm{E}_A(x_n))1\|_2 = 0$. This implies that $\lim_n \|x_n - \tau(x_n)1\|_2 = 0$ and so $(x_n)_n$ is trivial. Therefore, M is full. ☐

Let us point out that Choda's argument relies in a crucial way on the invariance of the probablity measure μ and no longer works when the action does not preserve a probability measure. We next investigate a large class of non-inner amenable groups for which one can prove the fullness property of the group measure space factors arising from arbitrary strongly ergodic actions.

Following [Oz03, Oz04, BO08], a countable group Γ is *bi-exact* if Γ is exact and if the following condition is satisfied:

$$\exists \mu : \Gamma \to \mathrm{Prob}(\Gamma) : x \mapsto \mu_x, \quad \forall g, h \in \Gamma, \quad \lim_{x \to \infty} \|\mu_{gxh} - g_*\mu_x\|_{\ell^1(\Gamma)} = 0. \tag{2.1}$$

Here and in what follows, we denote by $\mathrm{Prob}(\Gamma) = \{\eta \in \ell^1(\Gamma) \mid \eta \geq 0 \text{ and } \|\eta\|_{\ell^1(\Gamma)} = 1\}$. The class of bi-exact groups includes amenable groups, free groups [AO74], Gromov word-hyperbolic groups [Oz03] and discrete subgroups of connected simple Lie groups of real rank one [Sk88]. The following proposition shows that free groups satisfy (2.1).

Proposition 2.6. *For every $n \geq 2$, the free group $\mathbf{F}_n$ satisfies* (2.1).

Proof. Let $n \geq 2$ and put $\Gamma = \mathbf{F}_n = \langle a_1, \dots, a_n \rangle$. Denote by $|\cdot| : \Gamma \to \mathbf{N}$ the length function with respect to the generating set $S = \{a_1^{\pm 1}, \dots, a_n^{\pm 1}\}$. For every $g \in \Gamma$, denote by $(g_0, \dots, g_k)$ the unique geodesic path in the Cayley graph of (Γ, S) such that $g_0 = e$ and $g_k = g$ where $k = |g|$. Put $\mu_g = \frac{1}{k} \sum_{j=1}^{k} \delta_{g_j}$. For all $g, h \in \Gamma$, it is easy to see that

$$\begin{aligned} \|\mu_{gxh} - g_*\mu_x\|_{\ell^1(\Gamma)} &\leq \|\mu_{gxh} - g_*\mu_{xh}\|_{\ell^1(\Gamma)} + \|g_*\mu_{xh} - g_*\mu_x\|_{\ell^1(\Gamma)} \\ &= \|\mu_{gxh} - g_*\mu_{xh}\|_{\ell^1(\Gamma)} + \|\mu_{xh} - \mu_x\|_{\ell^1(\Gamma)} \\ &\leq \frac{|g|}{|gxh|} + \left| \frac{|xh|}{|gxh|} - 1 \right| + \left| \frac{|x|}{|xh|} - 1 \right| + \frac{|h|}{|x|} \to 0 \quad \text{as} \quad x \to \infty. \end{aligned}$$

This finishes the proof. ∎

The next result due to Houdayer–Isono (see [HI15b, Theorem C]) is a strengthening of Choda's result for arbitrary strongly ergodic actions of bi-exact groups.

Theorem 2.7. *Let Γ be any countable group satisfying* (2.1)*, (X, μ) any diffuse standard probability space and $\Gamma \curvearrowright (X, \mu)$ any ergodic free nonsingular action. We have the following dichotomy:*

- *Either $\Gamma \curvearrowright (X, \mu)$ is amenable.*
- *Or for every centralizing uniformly bounded sequence $(w_n)_n \in \ell^\infty(\mathbf{N}, M)$, we have $w_n - \mathrm{E}_{\mathrm{L}^\infty(X)}(w_n) \to 0$ strongly.*

In particular, if $\Gamma \curvearrowright (X, \mu)$ is strongly ergodic, then $\mathrm{L}(\Gamma \curvearrowright X)$ is a full factor.

Proof. Put $A = \mathrm{L}^\infty(X)$ and $M = \mathrm{L}(\Gamma \curvearrowright X)$. Denote by $\mathrm{E}_A : M \to A$ the canonical faithful normal conditional expectation and by $\varphi \in M_*$ the unique faithful state such that $\varphi \circ \mathrm{E}_A = \varphi$ and $\varphi|_A = \int_X \cdot \, \mathrm{d}\mu$. We simply write $\tau = \varphi|_A \in A_*$. Assume that there exists a centralizing uniformly bounded sequence $(w_n)_n \in \ell^\infty(\mathbf{N}, M)$ such that $\limsup_n \|w_n - \mathrm{E}_A(w_n)\|_\varphi > 0$. Up to extracting a subsequence, rescaling and replacing each w_n by $w_n - \mathrm{E}_A(w_n)$, we may further assume that for every $n \in \mathbf{N}$, we have $\|w_n\|_\varphi = 1$ and $\mathrm{E}_A(w_n) = 0$. We prove that $\Gamma \curvearrowright (X, \mu)$ is amenable following an idea due to Ozawa (see [Oz16, Example 8]). For every $n \in \mathbf{N}$, write $w_n = \sum_{h \in \Gamma} (w_n)^h u_h$ for the Fourier expansion of $w_n \in M$.

Claim. For every $h \in \Gamma$, we have $\lim_n \|(w_n)^h\|_2 = 0$.

By assumption, we already know that $(w_n)^e = \mathrm{E}_A(w_n) = 0$ for every $n \in \mathbf{N}$. Next assume that $h \in \Gamma \setminus \{e\}$. Since $(w_n)_n$ is centralizing, we have $\lim_n \|a(w_n)^h - (w_n)^h \sigma_h(a)\|_2 = 0$. Let $\omega \in \beta(\mathbf{N}) \setminus \mathbf{N}$ be any nonprincipal ultrafilter. Write $b_h = \sigma\text{-weak} \lim_{n \to \omega} ((w_n)^h)^* (w_n)^h \in A$. Since A is abelian, we obtain $ab_h = \sigma_h(a) b_h$ for every $a \in A$. Since $\Gamma \curvearrowright (X, \mu)$ is free, we have $b_h = 0$. This implies that $\lim_{n \to \omega} \|(w_n)^h\|_2 = 0$. Since this holds true for every $\omega \in \beta(\mathbf{N}) \setminus \mathbf{N}$, it follows that $\lim_n \|(w_n)^h\|_2 = 0$. This finishes the proof of the claim.

Choose a map $\mu : \Gamma \to \mathrm{Prob}(\Gamma)$ for which (2.1) is satisfied. For every $n \in \mathbf{N}$, define the positive linear mapping $\Phi_n : \mathrm{L}^\infty(\Gamma \times X) \to \mathrm{L}^\infty(X)$ by the formula

$$\forall f \in \mathrm{L}^\infty(\Gamma \times X), \quad \Phi_n(f) : x \mapsto \Phi_n(f)(x) = \sum_{g,h\in\Gamma} |(w_n)^g(x)|^2 \mu_g(h) f(h, x).$$

Observe that $(\Phi_n(f))_n$ is uniformly bounded for every $f \in \mathrm{L}^\infty(X)$ since $\sum_{g\in\Gamma} |(w_n)^g|^2 = \mathrm{E}_A(w_n^* w_n) \leq \sup_n \|w_n\|_\infty^2 < +\infty$ Next, fix a nonprincipal ultrafilter $\omega \in \beta(\mathbf{N}) \setminus \mathbf{N}$ and define the positive linear mapping $\Phi_\omega : \mathrm{L}^\infty(\Gamma \times X) \to \mathrm{L}^\infty(X)$ by the formula

$$\forall f \in \mathrm{L}^\infty(\Gamma \times X), \quad \Phi_\omega(f) = \sigma\text{-weak} \lim_{n\to\omega} \Phi_n(f).$$

Note that $\sigma\text{-weak} \lim_{n\to\omega} \mathrm{E}_A(w_n^* w_n) \in A^\Gamma = \mathbf{C}1$, $\lim_{n\to\omega} \tau(\mathrm{E}_A(w_n^* w_n)) = \lim_{n\to\omega} \|w_n\|_\varphi^2 = 1$ and thus $\sigma\text{-weak} \lim_{n\to\omega} \mathrm{E}_A(w_n^* w_n) = 1$. Then $\Phi_\omega(f) = f$ for every $f \in \mathrm{L}^\infty(X)$ and so $\Phi_\omega : \mathrm{L}^\infty(\Gamma \times X) \to \mathrm{L}^\infty(X)$ is a conditional expectation.

Claim. The conditional expectation $\Phi_\omega : \mathrm{L}^\infty(\Gamma \times X) \to \mathrm{L}^\infty(X)$ is Γ-equivariant.

Indeed, let $s \in \Gamma$ and $f \in \mathrm{L}^\infty(\Gamma \times X)$. On the one hand, we have

$$\begin{aligned}
\Phi_n(s \cdot f)(x) &= \sum_{g,h\in\Gamma} |(w_n)^g(x)|^2 \mu_g(h) f(s^{-1}h, s^{-1}x) \\
&= \sum_{g,h\in\Gamma} |(w_n)^g(x)|^2 \mu_g(sh) f(h, s^{-1}x) \\
&= \sum_{g,h\in\Gamma} |(w_n)^g(x)|^2 \mu_{s^{-1}gs}(h) f(h, s^{-1}x) \\
&\quad + \sum_{g,h\in\Gamma} |(w_n)^g(x)|^2 (s^{-1}{}_*\mu_g - \mu_{s^{-1}gs})(h) f(h, s^{-1}x) \\
&= \sum_{g,h\in\Gamma} |(w_n)^{sgs^{-1}}(x)|^2 \mu_g(h) f(h, s^{-1}x) \\
&\quad + \sum_{g,h\in\Gamma} |(w_n)^g(x)|^2 (s^{-1}{}_*\mu_g - \mu_{s^{-1}gs})(h) f(h, s^{-1}x)
\end{aligned}$$

On the other hand, we have

$$(s \cdot \Phi_n(f))(x) = \Phi_n(f)(s^{-1}x) = \sum_{g,h\in\Gamma} |(w_n)^g(s^{-1}x)|^2 \mu_g(h) f(h, s^{-1}x).$$

This implies that

$$\begin{aligned}
\|\Phi_n(s \cdot f) - s \cdot \Phi_n(f)\|_1 &\leq \|f\|_\infty \sum_{g\in\Gamma} \left\| |(w_n)^{sgs^{-1}}|^2 - s \cdot |(w_n)^g|^2 \right\|_1 \\
&\quad + \|f\|_\infty \sum_{g\in\Gamma} \|(w_n)^g\|_2^2 \cdot \|s^{-1}{}_*\mu_g - \mu_{s^{-1}gs}\|_{\ell^1(\Gamma)}.
\end{aligned} \tag{2.2}$$

Regarding the first term on the right hand side of (2.2), using Cauchy–Schwarz inequality and the fact that $||z_1| - |z_2|| \leq |z_1 - z_2|$ for all $z_1, z_2 \in \mathbf{C}$, we have

$$\begin{aligned}
&\sum_{g\in\Gamma} \left\| |(w_n)^{sgs^{-1}}|^2 - s \cdot |(w_n)^g|^2 \right\|_1 \\
&\leq \left(\sum_{g\in\Gamma} \left\| (w_n)^{sgs^{-1}} - s \cdot (w_n)^g \right\|_2^2 \right)^{1/2} \cdot \left(\sum_{g\in\Gamma} \left\| |(w_n)^{sgs^{-1}}| + s \cdot |(w_n)^g| \right\|_2^2 \right)^{1/2} \\
&\leq 2\|w_n\|_\infty \cdot \|(w_n - u_s w_n u_s^*)(\mathbf{1}_X \otimes \delta_e)\|.
\end{aligned}$$

Since $(w_n)_n$ is centralizing, it follows that

$$\lim_n \sum_{g\in\Gamma} \left\| |(w_n)^{sgs^{-1}}|^2 - s\cdot |(w_n)^g|^2 \right\|_1 = 0. \tag{2.3}$$

Regarding the second term on the right hand side of (2.2), using (2.1), for every $\varepsilon > 0$, there exists a finite subset $F \subset \Gamma$ such that for every $g \in \Gamma \setminus F$, we have $\|s^{-1}{}_*\mu_g - \mu_{s^{-1}gs}\|_{\ell^1(\Gamma)} \leq \varepsilon$. Using the previous claim, we have $\lim_n \|(w_n)^g\|_2^2 = 0$ for every $g \in F$. These two observations lead to

$$\limsup_n \sum_{g\in\Gamma} \|(w_n)^g\|_2^2 \cdot \|s^{-1}{}_*\mu_g - \mu_{s^{-1}gs}\|_{\ell^1(\Gamma)} \leq \varepsilon \limsup_n \sum_{g\in\Gamma\setminus F} \|(w_n)^g\|_2^2 \leq \varepsilon \limsup_n \|w_n\|_\varphi^2 \leq \varepsilon.$$

Since this holds true for every $\varepsilon > 0$, this leads to

$$\lim_n \sum_{g\in\Gamma} \|(w_n)^g\|_2^2 \cdot \|s^{-1}{}_*\mu_g - \mu_{s^{-1}gs}\|_{\ell^1(\Gamma)} = 0. \tag{2.4}$$

Combining (2.3) and (2.4), we obtain $\lim_n \|\Phi_n(s\cdot f) - s\cdot \Phi_n(f)\|_1 = 0$. Since the sequence $(\Phi_n(a))_n$ is uniformly bounded for every $a \in A$, this implies that $\Phi_\omega(s\cdot f) = s\cdot\Phi_\omega(f)$ and finishes the proof of the Claim.

Then the Γ-equivariant conditional expectation $\Phi_\omega : \mathrm{L}^\infty(\Gamma\times X) \to \mathrm{L}^\infty(X)$ witnesses the fact that the action $\Gamma \curvearrowright (X,\mu)$ is amenable.

We finally point out that thanks to the work of Connes and Haagerup on the classification of amenable factors ([Co75b, Co85, Ha85]), we do have a dichotomy in the statement of the theorem. □

We observe that using Theorem 2.7 and Vaes–Wahl results [VW17], we obtain full group measure space type III factors arising from strongly ergodic actions of the free groups with prescribed Connes' invariants [Co74, HMV17].

3. Lecture 3: Spectral gap for full factors

We first review Connes' joint distribution trick and we prove some useful technical results regarding norm estimates of certain commutators between elements in the von Neumann algebra and elements in its standard form. We then state Connes' spectral gap result for full factors of type II_1 and Marrakchi's spectral gap result for full factors of type III and derive some interesting consequences. We finally give a proof of Connes' spectral gap result due to Marrakchi and we prove Marrakchi's spectral gap result.

Preliminaries. Let M be any von Neumann algebra and denote by $(M, \mathrm{L}^2(M), J, \mathrm{L}^2(M)_+)$ its standard form. We first recall a useful technical tool known as Connes' joint distribution trick (See [Co75b, Proposition I.1] and [Ha85, Lemma 2.11]). Denote by $\mathcal{L}^\infty(B)$ the space of bounded Borel functions defined on a standard Borel space B.

Lemma 3.1. *Let $\zeta \in \mathrm{L}^2(M)$ and $x \in M_h$. Then there exists a finite positive Borel measure ν on $\mathbf{R}^2$ with $\operatorname{supp}(\nu) \subset \mathrm{Sp}(x) \times \mathrm{Sp}(x)$ such that*

$$\forall f, g \in \mathcal{L}^\infty(\mathbf{R}), \quad \|f(x)\zeta - \zeta g(x)\|^2 = \int_{\mathbf{R}^2} |f(s) - g(t)|^2 \, \mathrm{d}\nu(s,t). \tag{3.1}$$

Proof. Since $\mathrm{C}(\mathrm{Sp}(x) \times \mathrm{Sp}(x)) = \mathrm{C}(\mathrm{Sp}(x)) \otimes_{\max} \mathrm{C}(\mathrm{Sp}(x))$, there exists a unital $*$-representation $\pi : \mathrm{C}(\mathrm{Sp}(x) \times \mathrm{Sp}(x)) \to \mathbf{B}(\mathrm{L}^2(M))$ such that

$$\forall f, g \in \mathrm{C}(\mathrm{Sp}(x)), \quad \pi(f \otimes g)\zeta = f(x)\zeta g(x).$$

Denote by ν the finite positive Borel measure on $\mathrm{Sp}(x) \times \mathrm{Sp}(x)$ such that

$$\forall h \in \mathrm{C}(\mathrm{Sp}(x) \times \mathrm{Sp}(x)), \quad \langle \pi(h)\zeta, \zeta\rangle = \int_{\mathrm{Sp}(x)\times\mathrm{Sp}(x)} h(s,t) \, \mathrm{d}\nu(s,t). \tag{3.2}$$

By standard arguments, (3.2) holds for every $h \in \mathcal{L}^\infty(\mathrm{Sp}(x) \times \mathrm{Sp}(x))$. Then for all $f, g \in \mathcal{L}^\infty(\mathrm{Sp}(x))$, we have

$$\begin{aligned} \|f(x)\zeta - \zeta g(x)\|^2 &= \langle |f|^2(x)\zeta, \zeta\rangle + \langle \zeta |g|^2(x), \zeta\rangle - 2\Re\langle f(x)\zeta\overline{g}(x), \zeta\rangle \\ &= \int_{\mathrm{Sp}(x)\times\mathrm{Sp}(x)} \left(|f(s)|^2 + |g(t)|^2 - 2\Re(f(s)\overline{g(t)})\right) \mathrm{d}\nu(s,t) \\ &= \int_{\mathrm{Sp}(x)\times\mathrm{Sp}(x)} |f(s) - g(t)|^2 \, \mathrm{d}\nu(s,t). \end{aligned}$$

We may extend ν to a finite positive Borel measure on $\mathbf{R}^2$ by letting $\nu(\mathbf{R}^2 \setminus \mathrm{Sp}(x) \times \mathrm{Sp}(x)) = 0$. This finishes the proof. □

Next, we collect results due to Connes–Størmer [CS78, Theorem 2] and Marrakchi [Ma17, Lemma 2.3]. For every $a > 0$, denote by $e_a = \mathbf{1}_{[a,+\infty)} \in \mathcal{L}^\infty(\mathbf{R})$.

Theorem 3.2. *Let $\zeta \in \mathrm{L}^2(M)$ such that $J\zeta = \zeta$, $\varphi \in (M_*)_+$ and $x \in M_+$. Then we have*

$$\varphi(x) = \int_{\mathbf{R}_+^*} \varphi(e_a(x)) \, \mathrm{d}a \tag{3.3}$$

$$\|x - \varphi(x)1\|_\varphi^2 \leq \int_{\mathbf{R}_+^*} \varphi(e_a(x^2))(1 - \varphi(e_a(x^2))) \, \mathrm{d}a \tag{3.4}$$

$$\int_{\mathbf{R}_+^*} \|e_a(x^2)\zeta - \zeta e_a(x^2)\|^2 \, \mathrm{d}a \leq 2\|x\zeta\| \cdot \|x\zeta - \zeta x\|. \tag{3.5}$$

Proof. Fix $\zeta \in \mathrm{L}^2(M)$ such that $J\zeta = \zeta$, $\varphi \in (M_*)_+$ and $x \in M_+$.

Proof of (3.3)**.** Since $\varphi \in (M_*)_+$, we have

$$\varphi(x) = \varphi\left(\int_{\mathbf{R}_+^*} e_a(x)\, \mathrm{d}a\right) = \int_{\mathbf{R}_+^*} \varphi(e_a(x))\, \mathrm{d}a.$$

Proof of (3.4)**.** Observe that $\|x - \varphi(x)1\|_\varphi^2 = \varphi(x^2) - \varphi(x)^2$. By (3.3), we already know that

$$\varphi(x^2) = \varphi\left(\int_{\mathbf{R}_+^*} e_a(x^2)\, \mathrm{d}a\right) = \int_{\mathbf{R}_+^*} \varphi(e_a(x^2))\, \mathrm{d}a.$$

In order to prove (3.5), it suffices to show that

$$\int_{\mathbf{R}_+^*} \varphi(e_a(x^2))^2\, \mathrm{d}a \leq \varphi(x)^2.$$

In $M \mathbin{\overline{\otimes}} M$, we have $e_a(x^2) \otimes e_a(x^2) \leq e_a(x \otimes x)$. Applying (3.3) to $x \otimes x \in M \mathbin{\overline{\otimes}} M$ with $\varphi \otimes \varphi \in (M \mathbin{\overline{\otimes}} M)_*$, we have

$$\begin{aligned}
\int_{\mathbf{R}_+^*} \varphi(e_a(x^2))^2\, \mathrm{d}a &= \int_{\mathbf{R}_+^*} (\varphi \otimes \varphi)(e_a(x^2) \otimes e_a(x^2))\, \mathrm{d}a \\
&\leq \int_{\mathbf{R}_+^*} (\varphi \otimes \varphi)(e_a(x \otimes x))\, \mathrm{d}a \\
&= (\varphi \otimes \varphi)(x \otimes x) = \varphi(x)^2.
\end{aligned}$$

Proof of (3.5)**.** By Lemma 3.1, there exists a finite positive Borel measure ν on $\mathbf{R}^2$ with $\mathrm{supp}(\nu) \subset \mathrm{Sp}(x) \times \mathrm{Sp}(x) \subset \mathbf{R}_+$ such that

$$\forall a > 0, \quad \|e_a(x^2)\zeta - \zeta e_a(x^2)\|^2 = \int_{\mathbf{R}_+^2} |e_a(s^2) - e_a(t^2)|^2\, \mathrm{d}\nu(s,t).$$

Using Fubini's theorem, Cauchy–Schwarz inequality and Lemma 3.1, we have

$$\begin{aligned}
\int_{\mathbf{R}_+^*} \|e_a(x^2)\zeta - \zeta e_a(x^2)\|^2\, \mathrm{d}a &= \int_{\mathbf{R}_+^2} \left(\int_{\mathbf{R}_+^*} |e_a(s^2) - e_a(t^2)|^2\, \mathrm{d}a\right) \mathrm{d}\nu(s,t) \\
&= \int_{\mathbf{R}_+^2} |s-t| \cdot (s+t)\, \mathrm{d}\nu(s,t) \\
&\leq \left(\int_{\mathbf{R}_+^2} |s-t|^2\, \mathrm{d}\nu(s,t)\right)^{1/2} \cdot \left(\int_{\mathbf{R}_+^2} (s+t)^2\, \mathrm{d}\nu(s,t)\right)^{1/2} \\
&\leq \left(\int_{\mathbf{R}_+^2} |s-t|^2\, \mathrm{d}\nu(s,t)\right)^{1/2} \cdot \left(2\int_{\mathbf{R}_+^2} (s^2+t^2)\, \mathrm{d}\nu(s,t)\right)^{1/2} \\
&= \|x\zeta - \zeta x\| \cdot \left(2\|x\zeta\|^2 + 2\|\zeta x\|^2\right)^{1/2} \\
&= 2\|x\zeta\| \cdot \|x\zeta - \zeta x\|.
\end{aligned}$$

This finishes the proof. ◻

Spectral gap property of full factors. As we have seen in Theorem 1.5, any non-inner amenable countable group Γ gives rise to a full type II_1 factor $M = \mathrm{L}(\Gamma)$ for which there exist some $\kappa > 0$ and a finite critical set $\{g_1, \dots, g_k\} \subset \Gamma$ witnessing spectral gap for the conjugation representation $\mathrm{Ad}^0 : \Gamma \to \mathcal{U}(\mathrm{L}^2(\Gamma) \ominus \mathbf{C}\delta_e)$. In other words, the constant $\kappa > 0$ and the critical set $\{u_{g_1}, \dots, u_{g_k}\} \subset \mathcal{U}(M)$ witnesses spectral gap for the conjugation representation $\mathcal{U}(M) \to \mathcal{U}(\mathrm{L}^2(M) \ominus \mathbf{C}\xi_\tau)$. Connes' celebrated result [Co75b, Theorem 2.1] shows that such a spectral gap property holds for *every* full factor of type II_1 (with separable predual). Connes' spectral gap theorem is a remarkable result that played a key role in Connes' proof of the uniqueness of the amenable type II_1 factor [Co75b]. As we will see in Lecture 4, it has also been an important tool in Popa's deformation/rigidity theory.

We state and we will give a complete proof of the following more general version of [Co75b, Theorem 2.1] without assuming separability of the predual.

Theorem 3.3. *Let M be any type II_1 factor. Then the following conditions are equivalent:*

(i) *M is a full factor.*
(ii) *There exist $\kappa > 0$ and $u_1, \dots, u_k \in \mathcal{U}(M)$ such that*

$$\forall x \in M, \quad \|x - \tau(x)1\|_2^2 \leq \kappa \sum_{j=1}^{k} \|u_j x - x u_j\|_2^2. \tag{3.6}$$

Note that (3.6) exactly means that the conjugation representation

$$\mathcal{U}(M) \to \mathcal{U}(\mathrm{L}^2(M) \ominus \mathbf{C}\xi_\tau) : \eta \mapsto uJuJ\eta$$

has spectral gap (here we regard $\mathcal{U}(M)$ as a discrete group).

It is easy to see that for any type II_1 factor M, if $P_{\mathbf{C}\xi_\tau} \in \mathrm{C}^*(M, JMJ)$, then M is full. Theorem 3.3 shows that the converse holds true.

Corollary 3.4. *Let M be any full factor of type II_1. Then $\mathbf{K}(\mathrm{L}^2(M)) \subset \mathrm{C}^*(M, JMJ)$.*

Proof. Since M is a full factor of type II_1, we may choose $\kappa > 0$ and $u_1, \dots, u_k \in \mathcal{U}(M)$ witnessing (3.6). Then the positive selfadjoint bounded operator $T = \sum_{j=1}^k |u_j - Ju_jJ|^2$ on $\mathrm{L}^2(M)$ has its spectrum $\sigma(T)$ contained in $\{0\} \cup [\frac{1}{\kappa}, +\infty)$ and 0 is an eigenvalue for T with multiplicity 1. Since $T \in \mathrm{C}^*(M, JMJ)$ and since $\mathbf{1}_{\{0\}}$ is a continuous function on $\sigma(T)$, it follows that $P_{\mathbf{C}\xi_\tau} = \mathbf{1}_{\{0\}}(T) \in \mathrm{C}^*(M, JMJ)$. Since ξ_τ is M-cyclic and since $\mathrm{C}^*(M, JMJ)$ is $\|\cdot\|_\infty$-closed, we infer that $\mathbf{K}(\mathrm{L}^2(M)) \subset \mathrm{C}^*(M, JMJ)$. ☐

Corollary 3.4 implies that the tensor product of any two full factors of type II_1 is still full. Indeed, let (M_1, τ_1) and (M_2, τ_2) by any full factors of type II_1. Put $M = M_1 \mathbin{\overline{\otimes}} M_2$, $\tau = \tau_1 \otimes \tau_2$ and $J = J_1 \otimes J_2$. By Corollary 3.4, we have $\mathbf{K}(\mathrm{L}^2(M_1)) \subset \mathrm{C}^*(M_1, J_1M_1J_1)$ and $\mathbf{K}(\mathrm{L}^2(M_2)) \subset \mathrm{C}^*(M_2, J_2M_2J_2)$. Since $\mathrm{C}^*(M_1, J_1M_1J_1) \otimes \mathbf{C}1 \subset \mathrm{C}^*(M, JMJ)$ and $\mathbf{C}1 \otimes \mathrm{C}^*(M_2, J_2M_2J_2) \subset \mathrm{C}^*(M, JMJ)$, we have $P_{\mathbf{C}\xi_\tau} = P_{\mathbf{C}\xi_{\tau_1}} \otimes P_{\mathbf{C}\xi_{\tau_2}} \in \mathrm{C}^*(M, JMJ)$. This implies that M is full.

Marrakchi recently obtained in [Ma16, Theorem A] an analogue of Connes' spectral gap theorem in the setting of type III factors. We only state his result for σ-finite factors and refer the reader to [Ma16, Theorem A] for a more general statement that holds for arbitrary factors.

Theorem 3.5. *Let M be any σ-finite factor of type* III. *The following conditions are equivalent:*

(i) *M is a full factor.*
(ii) *There exist $\kappa > 0$, a faithful state $\varphi \in M_*$ and $\xi_1, \dots, \xi_k \in \mathrm{Ball}(M)\xi_\varphi$ such that $J\xi_j = \xi_j$ for every $1 \leq j \leq k$ and such that*

$$\forall x \in M, \quad \|x - \varphi(x)1\|_\varphi^2 \leq \kappa \sum_{j=1}^{k} \|x\xi_j - \xi_j x\|^2. \tag{3.7}$$

Let us point out that the spectral gap estimates (3.6) in Theorem 3.3 and (3.7) in Theorem 3.5 are slightly different. Indeed, for every $1 \leq j \leq k$ write $\xi_j = a_j \xi_\varphi$ where $a_j \in \mathrm{Ball}(M)$. By the parallellogram inequality, for every $1 \leq j \leq k$ and every $x \in M$, since $x\xi_j - \xi_j x = (xa_j - a_j x)\xi_\varphi + a_j(x\xi_\varphi - \xi_\varphi x)$, we have

$$\|x\xi_j - \xi_j x\|^2 \leq 2\|xa_j - a_j x\|_\varphi^2 + 2\|x\xi_\varphi - \xi_\varphi x\|^2.$$

Then (3.7) becomes

$$\forall x \in M, \quad \|x - \varphi(x)1\|_\varphi^2 \leq 2\kappa \sum_{j=1}^k \|xa_j - a_j x\|_\varphi^2 + 2k\kappa \|x\xi_\varphi - \xi_\varphi x\|^2. \tag{3.8}$$

We can now see that the main difference between (3.6) and (3.8) is the existence of the extra term $2k\kappa\|x\xi_\varphi - \xi_\varphi x\|^2$ on the right hand side of the inequality which measures "how far x commutes with ξ_φ". For that reason, we cannot obtain the same conclusion as the one in Corollary 3.4.

In [HMV16], Houdayer–Marrakchi–Verraedt obtained a strengthening of (3.8) and proved that fullness is stable under taking tensor product. In [HMV17], Houdayer–Marrakchi–Verraedt also obtained a spectral gap characterization of strong ergodicity for arbitrary nonsingular actions. Since these results are beyond the scope of these notes, we will not dwell further on that.

Proof of the spectral gap property of full factors. We give a simple proof of Connes' spectral gap theorem for full factors of type II_1 (see Theorem 3.3) due to Marrakchi [Ma17]. We then prove Marrakchi's spectral gap theorem for full factors of type III (see Theorem 3.5). Let us point out that a simple proof of Connes' spectral gap theorem for full factors of type II_1 is also given in the forthcoming book [AP18].

The following crucial result is due to Marrakchi [Ma17, Proposition 2.2].

Theorem 3.6. *Let M be any σ-finite von Neumann algebra, $\varphi \in M_*$ any faithful state and $\xi_1, \dots, \xi_k \in \mathrm{Ball}(M)\xi_\varphi$ any elements such that $J\xi_j = \xi_j$ for every $1 \leq j \leq k$. The following assertions are equivalent:*

(i) *There exists $\kappa_1 > 0$ such that for all projections $p \in M$, we have*

$$\varphi(p)(1 - \varphi(p)) \leq \kappa_1 \sum_{j=1}^k \|p\xi_j - \xi_j p\|^2. \tag{3.9}$$

(ii) *There exists $\kappa_2 > 0$ such that for all elements $x \in M$, we have*

$$\|x - \varphi(x)1\|_\varphi^2 \leq \kappa_2 \sum_{j=1}^k \|x\xi_j - \xi_j x\|^2. \tag{3.10}$$

Proof. (ii) $\Rightarrow$ (i) Applying (3.10) with $\kappa_1 = \kappa_2$ to $x = p$, we obtain (3.9).

(i) $\Rightarrow$ (ii) First, let $x \in M_+$ be any positive element. For every $1 \leq j \leq k$, write $\xi_j = a_j\xi_\varphi = \xi_\varphi a_j^*$ where $a_j \in \mathrm{Ball}(M)$. Then we have $\|x\xi_j\| = \|x\xi_\varphi a_j^*\| \leq \|x\xi_\varphi\|$. Using (3.4) and (3.5) in

Theorem 3.2 and using the assumption, we obtain

$$\begin{aligned}\|x-\varphi(x)1\|_\varphi^2 &\leq \int_{\mathbf{R}_+^*} \varphi(e_a(x^2))(1-\varphi(e_a(x^2)))\,\mathrm{d}a \\ &\leq \kappa_1 \sum_{j=1}^k \int_{\mathbf{R}_+^*} \|e_a(x^2)\xi_j - \xi_j e_a(x^2)\|^2\,\mathrm{d}a \\ &\leq 2\kappa_1 \sum_{j=1}^k \|x\xi_j\|\cdot\|x\xi_j - \xi_j x\| \\ &\leq 2\kappa_1 \sum_{j=1}^k \|x\|_\varphi \cdot \|x\xi_j - \xi_j x\|.\end{aligned}$$

Next, let $x \in M_h$ be any selfadjoint element such that $\varphi(x) = 0$. Write $x = x_+ - x_-$ where $x_+, x_- \in M_+$ and $x_+ x_- = 0$. Using the parallellogram inequality and the above inequalities for x_+ and x_-, we obtain

$$\begin{aligned}\|x\|_\varphi^2 &\leq 2\left(\|x_+ - \varphi(x_+)1\|_\varphi^2 + \|x_- - \varphi(x_-)1\|_\varphi^2\right) \\ &\leq 4\kappa_1 \sum_{j=1}^k \left(\|x_+\|_\varphi \cdot \|x_+\xi_j - \xi_j x_+\| + \|x_-\|_\varphi \cdot \|x_-\xi_j - \xi_j x_-\|\right).\end{aligned} \tag{3.11}$$

Since $x = x_+ - x_-$ and $x_+ x_- = 0$, we have $\|x\|_\varphi^2 = \|x_+\|_\varphi^2 + \|x_-\|_\varphi^2$. Thus, we have $\|x_\pm\|_\varphi \leq \|x\|_\varphi$. For every $1 \leq j \leq k$, we moreover have

$$\begin{aligned}\|x\xi_j - \xi_j x\|^2 &= \|x_+\xi_j - \xi_j x_+\|^2 + \|x_-\xi_j - \xi_j x_-\|^2 - 2\Re\langle x_+\xi_j - \xi_j x_+, x_-\xi_j - \xi_j x_-\rangle \\ &= \|x_+\xi_j - \xi_j x_+\|^2 + \|x_-\xi_j - \xi_j x_-\|^2 + 2\Re\langle x_+\xi_j, \xi_j x_-\rangle + 2\Re\langle \xi_j x_+, x_-\xi_j\rangle \\ &\geq \|x_+\xi_j - \xi_j x_+\|^2 + \|x_-\xi_j - \xi_j x_-\|^2\end{aligned}$$

since $x_+ J x_- J \geq 0$ and $x_- J x_+ J \geq 0$. Thus, we have $\|x_\pm \xi_j - \xi_j x_\pm\| \leq \|x\xi_j - \xi_j x\|$. From (3.11), we obtain

$$\|x\|_\varphi^2 \leq 8\kappa_1 \|x\|_\varphi \sum_{j=1}^k \|x\xi_j - \xi_j x\|$$

and so

$$\|x\|_\varphi \leq 8\kappa_1 \sum_{j=1}^k \|x\xi_j - \xi_j x\|.$$

Using Cauchy–Schwarz inequality and letting $\kappa_2 = (8\kappa_1)^2 k$, we obtain

$$\|x\|_\varphi^2 \leq \kappa_2 \sum_{j=1}^k \|x\xi_j - \xi_j x\|^2. \tag{3.12}$$

Next, let $x \in M_h$ be any selfadjoint element. Applying (3.12) to $x - \varphi(x)1$, we obtain (3.10) for $x \in M_h$.

Finally, let $x \in M$ be any element. Writing $x = \Re(x) + \mathrm{i}\Im(x)$, applying (3.10) to $\Re(x)$ and $\Im(x)$ respectively, using the fact that $J\xi_j = \xi_j$ for every $1 \leq j \leq k$ and using Pythagoras theorem, we obtain (3.10) for $x \in M$. □

Proof of Theorem 3.3. (ii) ⇒ (i) It is obvious.

(i) $\Rightarrow$ (ii) We prove the implication by contradiction. Assume that M satisfies (i) of Theorem 3.3 but does not satisfy (ii) of Theorem 3.3. Since M satisfies (i) of Theorem 3.3, for $\varepsilon = 1/4$, there exist $\alpha > 0$ and $b_1, \dots, b_k \in M_h$ such that

$$\forall p \in \mathcal{P}(M), \quad \sum_{j=1}^{k} \|pb_j - b_j p\|_2^2 \leq \alpha \quad \Rightarrow \quad \min(\tau(p), 1 - \tau(p)) \leq \frac{1}{4}. \tag{3.13}$$

We start by proving the following crucial claim.

Claim. For every nonzero projection $p \in M$, pMp does not satisfy (ii) of Theorem 3.3.

Proof of the Claim. Denote by I the directed set of all pairs (F, ε) where $F \subset \mathcal{U}(M)$ is a nonempty finite subset and $\varepsilon > 0$. Let $i = (F, \varepsilon) \in I$. Since M does not satisfy (ii) of Theorem 3.3, there exists $x_i \in M$ such that

$$\|x_i - \tau(x_i)1\|_2^2 > \frac{1}{\varepsilon} \sum_{u \in F} \|x_i u - u x_i\|_2^2.$$

In particular, $\|x_i - \tau(x_i)1\|_2 > 0$. Letting $y_i = \frac{1}{\|x_i - \tau(x_i)1\|_2}(x_i - \tau(x_i)1) \in M$, we have $\|y_i\|_2 = 1$, $\tau(y_i) = 0$ and

$$\sum_{u \in F} \|y_i u - u y_i\|_2^2 < \varepsilon.$$

Since any element in M is a linear combination of at most 4 elements in $\mathcal{U}(M)$, $(y_i)_{i \in I}$ is a central net in M, meaning that $\lim_i \|y_i b - b y_i\|_2 = 0$ for every $b \in M$. Note however that $(y_i)_{i \in I}$ need not be uniformly bounded.

Fix a nonzero projection $p \in M$. Observe that $(py_ip)_{i \in I}$ is a central net in pMp. Let ω be a cofinal ultrafilter on I. Denote by $\psi_\omega \in M^*$ the state defined by $\psi_\omega(x) = \lim_{i \to \omega} \tau(y_i^* x y_i)$ for all $x \in M$. Since $(y_i)_{i \in I}$ is a central net in M, ψ_ω is tracial on M and thus $\psi_\omega = \tau$ by uniqueness of the trace on M. This implies that $\lim_{i \to \omega} \|py_ip\|_2 = \|p\|_2$. Denote by $\zeta = \text{weak} \lim_{i \to \omega} y_i \xi_\tau \in \mathrm{L}^2(M)$. Since $\tau(y_i) = 0$ for every $i \in I$, we have $\langle \zeta, \xi_\tau \rangle = 0$. Since $(y_i)_{i \in I}$ is a central net in M, we have $b\zeta = \zeta b$ for every $b \in M$ and so $\zeta \in \mathbf{C}\xi_\tau$. Thus, we have $\zeta = 0$. This implies that $\lim_{i \to \omega} \tau(py_ip) = 0$. Thus, the net $(py_ip)_{i \in I}$ witnesses the fact that pMp does not satisfy (ii) of Theorem 3.3. □

Denote by J the directed set of all pairs (F, δ) where $F \subset M_h$ is a finite subset such that $\{b_1, \dots, b_k\} \subset F$ and $0 < \delta \leq \alpha$. Let $j = (F, \delta) \in J$. Denote by Λ_j the inductive set of all projections $q \in M$ such that $\tau(q) \leq 1/4$ and $\sum_{b \in F} \|qb - bq\|_2^2 \leq \delta \tau(q)$. By Zorn's lemma, we may choose a maximal projection $p = p_j \in \Lambda_j$. We claim that $\tau(p) = 1/4$. Assume that this is not the case and so $\tau(p) < 1/4$. By the previous claim, $p^\perp M p^\perp$ does not satisfy (ii) of Theorem 3.3. Put $\tau_{p^\perp} = \frac{1}{\tau(p^\perp)} \tau(p^\perp \cdot p^\perp)$. Then $p^\perp M p^\perp$ does not satisfy (ii) of Theorem 3.6 with respect to the tracial state $\varphi = \tau_{p^\perp}$ (recall that any element is a linear combination of at most four unitaries) and thus $p^\perp M p^\perp$ does not satisfy (i) of Theorem 3.6 either with respect to the tracial state $\varphi = \tau_{p^\perp}$. There exists a projection $q \in p^\perp M p^\perp$ such that $\tau(q)(\tau(p^\perp) - \tau(q)) > \frac{1}{\delta} \sum_{b \in F} \|q\, p^\perp b p^\perp - p^\perp b p^\perp\, q\|_2^2$. Up to replacing $q \in p^\perp M p^\perp$ by

$p^\perp - q \in p^\perp M p^\perp$, we may assume that $\tau(q) \leq 1/2$. We have

$$\begin{aligned}\sum_{b\in F} \|(p+q)b - b(p+q)\|_2^2 &= \sum_{b\in F} \|(pbq^\perp - q^\perp bp) + (q\,p^\perp bp^\perp - p^\perp bp^\perp\, q)\|_2^2 \\ &= \sum_{b\in F} \|q^\perp(pb - bp)q^\perp\|_2^2 + \sum_{b\in F} \|q\,p^\perp bp^\perp - p^\perp bp^\perp\, q\|_2^2 \\ &\leq \sum_{b\in F} \|pb - bp\|_2^2 + \sum_{b\in F} \|q\,p^\perp bp^\perp - p^\perp bp^\perp\, q\|_2^2 \\ &\leq \delta\tau(p) + \delta\tau(q) = \delta\tau(p+q).\end{aligned}$$

Since $p \in \Lambda_j$ is a maximal element, the above inequalities show that we have $\tau(p+q) > 1/4$. Since in particular we have $\sum_{j=1}^k \|(p+q)b_j - b_j(p+q)\|_2^2 \leq \alpha$ and since $\tau(p+q) > 1/4$, (3.13) implies that $\tau(p+q) \geq 3/4$. However, since $\tau(p+q) = \tau(p) + \tau(q) < 1/4 + 1/2 = 3/4$, we obtain a contradiction. Therefore, we have $\tau(p_j) = \tau(p) = 1/4$.

Thus, we have obtained a net of central projections $(p_j)_{j\in J}$ in M such that $\tau(p_j) = 1/4$ for every $j \in J$. This contradicts the fact that M satisfies (i) of Theorem 3.3. Therefore, we have proved that (i) ⇒ (ii). □

Proof of Theorem 3.5. (ii) ⇒ (i) It is obvious.

(i) ⇒ (ii) The proof of this implication is entirely analogous to the one of (i) ⇒ (ii) in Theorem 3.3. We proceed by contradiction and assume that M satisfies (i) of Theorem 3.5 but does not satisfy (ii) of Theorem 3.5. Fix a faithful state $\varphi \in M_*$. Since M satisfies (i) of Theorem 3.5 and since the linear span of all the vectors $\zeta \in \mathrm{Ball}(M)\xi_\varphi$ such that $J\zeta = \zeta$ is dense in $\mathrm{L}^2(M)$ (this follows from the fact that the set of all φ-analytic elements is $*$-strongly dense in M), for $\varepsilon = 1/4$, there exist $\alpha > 0$ and $\xi_1, \ldots, \xi_k \in \mathrm{Ball}(M)\xi_\varphi$ such that $J\xi_j = \xi_j$ for every $1 \leq j \leq k$ and such that

$$\forall p \in \mathcal{P}(M), \quad \sum_{j=1}^k \|p\xi_j - \xi_j p\|^2 \leq \alpha \quad \Rightarrow \quad \min(\varphi(p), 1-\varphi(p)) \leq \frac{1}{4}. \tag{3.14}$$

Denote by J the directed set of all pairs (F, δ) where $F \subset \mathrm{Ball}(M)\xi_\varphi$ is a finite subset such that $\{\xi_1, \ldots, \xi_k\} \subset F$ and $J\zeta = \zeta$ for all $\zeta \in F$ and $0 < \delta \leq \alpha$. Let $j = (F, \delta) \in J$. Denote by Λ_j the inductive set of all projections $q \in M$ such that $\varphi(q) \leq 1/4$ and $\sum_{\zeta\in F} \|q\zeta - \zeta q\|^2 \leq \delta\varphi(q)$. By Zorn's lemma, we may choose a maximal projection $p = p_j \in \Lambda_j$. We claim that $\varphi(p) = 1/4$. Assume that this is not the case and so $\varphi(p) < 1/4$. By assumption, $p^\perp M p^\perp \cong M$ does not satisfy (ii) of Theorem 3.5. Put $\varphi_{p^\perp} = \frac{1}{\varphi(p^\perp)}\varphi(p^\perp \cdot p^\perp) \in (p^\perp M p^\perp)_*$. Then $p^\perp M p^\perp$ does not satisfy (ii) of Theorem 3.6 with respect to the faithful state $\varphi_{p^\perp}$ and thus $p^\perp M p^\perp$ does not satisfy (i) of Theorem 3.6 either with respect to the faithful state $\varphi_{p^\perp}$. There exists a projection $q \in p^\perp M p^\perp$ such that $\varphi(q)(\varphi(p^\perp) - \varphi(q)) > \frac{1}{\delta}\sum_{\zeta\in F} \|q\,p^\perp \zeta p^\perp - p^\perp \zeta p^\perp\, q\|^2$. Up to replacing $q \in p^\perp M p^\perp$ by $p^\perp - q \in p^\perp M p^\perp$ we may assume that $\varphi(q) \leq 1/2$. We have

$$\sum_{\zeta\in F} \|(p+q)\zeta - \zeta(p+q)\|^2 \leq \sum_{\zeta\in F} \|p\zeta - \zeta p\|_2^2 + \sum_{\zeta\in F} \|q\,p^\perp \zeta p^\perp - p^\perp \zeta p^\perp\, q\|^2 \leq \delta\varphi(p+q).$$

Since $p \in \Lambda_j$ is a maximal element, the above inequality shows that we have $\varphi(p+q) > 1/4$. Since in particular we have $\sum_{j=1}^k \|(p+q)\xi_j - \xi_j(p+q)\|^2 \leq \alpha$ and since $\varphi(p+q) > 1/4$, (3.14) implies that $\varphi(p+q) \geq 3/4$. However, since $\varphi(p+q) = \varphi(p) + \varphi(q) < 1/4 + 1/2 = 3/4$, we obtain a contradiction. Therefore, we have $\varphi(p_j) = \varphi(p) = 1/4$.

Thus, we have obtained a net of centralizing projections $(p_j)_{j\in J}$ in M such that $\varphi(p_j) = 1/4$ for every $j \in J$. This contradicts the fact that M satisfies (i) of Theorem 3.5. Therefore, we have proved that (i) ⇒ (ii). □

4. Lecture 4: Unique McDuff decomposition

Popa's intertwining theory. We say that a von Neumann subalgebra $P \subset M$ is *with expectation* if there exists a faithful normal conditional expectation $\mathrm{E}_P : M \to P$. Recall that when M is tracial, any von Neumann subalgebra $P \subset M$ is with expectation.

Popa introduced his powerful *intertwining-by-bimodules* theory in [Po01, Po03] in the setting of tracial von Neumann algebras. Popa's intertwining theory has recently been extended to arbitrary von Neumann algebras in [BH16, HI15a]. Let M be any σ-finite von Neumann algebra and $P, Q \subset M$ any tracial von Neumann subalgebras with expectation. Following [Po01, Po03], we say that P *embeds into* Q *inside* M and write $P \preceq_M Q$ if there exist projections $p \in P$, $q \in Q$, a nonzero partial isometry $v \in pMq$ and a unital normal $*$-homomorphism $\pi : pPp \to qQq$ such that $xv = v\pi(x)$ for every $x \in pPp$. Observe that $vv^* \in (pPp)' \cap pMp$ and $v^*v \in \pi(pPp)' \cap qMq$.

The following criterion provides a useful analytical tool to exploit the condition $P \npreceq_M Q$. This result is essentially due to Popa [Po03, Theorem 2.1 and Corollary 2.3] (see also [HV12, Theorem 2.3]).

Theorem 4.1. *Let M be any σ-finite von Neumann algebra and $P, Q \subset M$ any tracial von Neumann subalgebras with expectation. The following assertions are equivalent:*

(i) $P \npreceq_M Q$.
(ii) *There is a net of unitaries $(u_i)_{i \in I}$ in P such that* $\lim_i \|\mathrm{E}_Q(x^* u_i y)\|_2 = 0$ *for all* $x, y \in M$.

Proof. (ii) $\Rightarrow$ (i) We prove this implication by contradiction. Take a net of unitaries $(u_i)_{i\in I}$ in P such that $\lim_i \|\mathrm{E}_Q(x^* u_i y)\|_2 = 0$ for all $x, y \in M$. Take (p, q, v, π) witnessing the fact that $P \preceq_M Q$. Up to shrinking $p \in P$ if necessary, we may assume that the normal $*$-homomorphism $pPp \to qMq : x \mapsto \pi(x)v^*v$ is injective. Since P is tracial, using [KR97, Proposition 8.2.1], we may choose nonzero partial isometries $w_1, \dots, w_k \in P$ such that $w_j^* w_j \leq p$ for every $1 \leq j \leq k$ and $\sum_{j=1}^k w_j w_j^* = z \in \mathcal{Z}(P)$. Note that for every $1 \leq j \leq k$, $w_j v \neq 0$ since $v^* w_j v = v^* v \pi(w_j) \neq 0$. Define the normal $*$-homomorphism $\Theta : Pz \to \mathbf{M}_k(qQq)$ by $\Theta(x) = [\pi(w_i^* x w_j)]_{ij}$ and the nonzero partial isometry $V = [w_1 v \cdots w_k v] \in \mathbf{M}_{1,k}(\mathbf{C}) \otimes pMq$. We have $xV = V\Theta(x)$ for all $x \in Pz$.

Observe that $\Theta(z) = \mathrm{Diag}(\pi(w_j^* w_j))$ and $V^*V = \mathrm{Diag}(v^* w_j^* w_j v) \leq \Theta(z)$. Denote by $\mathrm{E} : \Theta(z)\,\mathbf{M}_k(qMq)\,\Theta(z) \to \Theta(z)\,\mathbf{M}_k(qQq)\,\Theta(z)$ a faithful normal conditional expectation. Then for every $i \in I$, we have

$$\begin{aligned}\|\mathrm{E}(V^*V)\|_2 &= \|\mathrm{E}(V^*V)\Theta(u_i z)\|_2 \\ &= \|\mathrm{E}(V^*V\Theta(u_i z))\|_2 \\ &= \|\mathrm{E}(V^* u_i V)\|_2.\end{aligned}$$

Using the assumption, we have $\lim_i \|\mathrm{E}(V^* u_i V)\|_2 = 0$ and so $\mathrm{E}(V^*V) = 0$. This however contradicts the fact that $V \neq 0$.

(i) $\Rightarrow$ (ii) We prove this implication by contraposition. Fix a faithful normal conditional expectation $\mathrm{E}_Q : M \to Q$ and a faithful state $\varphi \in M_*$ such that $\varphi \circ \mathrm{E}_Q = \varphi$ and $\tau = \varphi|_Q$ is tracial. Denote by $(M, \mathrm{L}^2(M), J, \mathrm{L}^2(M)_+)$ the standard form of M. Denote by $\langle M, Q\rangle = (JQJ)' \cap \mathbf{B}(\mathrm{L}^2(M))$ Jones basic construction and by $e_Q : \mathrm{L}^2(M) \to \mathrm{L}^2(Q)$ Jones projection. Observe that $\langle M, Q\rangle = (M \cup \{e_Q\})''$. Since τ is tracial on Q, there exists a canonical faithful normal semifinite trace Tr on $\langle M, Q\rangle$ such that $\mathrm{Tr}(TT^*) = \tau(T^*T)$ for all right Q-modular bounded linear maps $T : \mathrm{L}^2(Q) \to \mathrm{L}^2(M)$. In particular, we have $\mathrm{Tr}(e_Q) = 1$.

Since (ii) does not hold, there exist $\varepsilon > 0$ and a finite subset $\mathcal{F} \subset M$ such that

$$\forall u \in \mathcal{U}(P), \quad \sum_{x,y \in \mathcal{F}} \|\mathrm{E}_Q(x^* u y)\|_2^2 \geq \varepsilon^2.$$

Put $d = \sum_{x\in\mathcal{F}} x e_Q x^* \in \langle M, Q\rangle_+$. We have $\mathrm{Tr}(d) = \sum_{x\in\mathcal{F}} \tau(\mathrm{E}_Q(x^*x)) < +\infty$. Moreover, for every $u \in \mathcal{U}(P)$, we have

$$(4.1)\qquad \sum_{y\in\mathcal{F}} \langle u^* du\, y\xi_\varphi, y\xi_\varphi\rangle = \sum_{x,y\in\mathcal{F}} \langle u^* x e_Q x^* u\, y\xi_\varphi, y\xi_\varphi\rangle = \sum_{x,y\in\mathcal{F}} \|\mathrm{E}_Q(x^*uy)\|_2^2 \geq \varepsilon^2.$$

Denote by $\mathcal{C}$ the σ-weak closure in $\langle M, Q\rangle$ of the convex hull of $\{u^*du : w \in \mathcal{U}(P)\}$. We have $0 \notin \mathcal{C}$ by (4.1). Since $\mathcal{C}$ is bounded both in $\|\cdot\|_\infty$ and $\|\cdot\|_{2,\mathrm{Tr}}$, $\mathcal{C}$ can be regarded as a closed bounded convex subset of $\mathrm{L}^2(\langle M, e_Q\rangle, \mathrm{Tr})$. Denote by $c \in \mathcal{C}$ the unique circumcenter of $\mathcal{C}$. For every $u \in \mathcal{U}(P)$, since $u^*\mathcal{C}u = \mathcal{C}$, we have $u^*cu = c$. Thus, $c \in P' \cap \langle M, e_Q\rangle_+$ such that $c \neq 0$ and $\mathrm{Tr}(c) < +\infty$ (this is due to the fact that Tr is σ-weakly lower semi-continuous).

Define the nonzero spectral projection $e = \mathbf{1}_{[\|c\|_\infty/2, \|c\|_\infty]}(c) \in P' \cap \langle M, Q\rangle_+$. Since $\frac{\|c\|_\infty}{2} e \leq c\, e$, we have $\mathrm{Tr}(e) < +\infty$. Let $\mathcal{H} = e\mathrm{L}^2(M)$. Then ${}_P\mathcal{H}_Q$ is a nonzero P-Q-subbimodule of ${}_P\mathrm{L}^2(M)_Q$ such that $\dim(\mathcal{H}_Q) = \mathrm{Tr}(e) < +\infty$. By [BO08, Proposition F.10], there exist a nonzero projection $p \in P$ and a nonzero pPp-Q-subbimodule $\mathcal{K} \subset p\mathcal{H}$ such that $\mathcal{K}$ is isomorphic as a right Q-module to a right Q-submodule of $\mathrm{L}^2(Q)_Q$. Denote by $V : \mathcal{K}_Q \to \mathrm{L}^2(Q)_Q$ the corresponding right Q-modular isometry. For every $x \in pPp$, since VxV^* commutes with the right Q-action on $\mathrm{L}^2(Q)$, we have $VxV^* \in qQq$ where $q = VV^*$. Therefore $\pi : pPp \to qQq : x \mapsto VxV^*$ is a unital normal $*$-homomorphism. Put $\xi = V^*\xi_\tau \in \mathcal{K}$. We have $\xi \neq 0$ since $V\xi = VV^*\xi_\tau = q\xi_\tau \neq 0$. Moreover, for every $x \in pPp$, we have

$$x\xi = xV^*\xi_\tau = V^*\pi(x)\xi_\tau = V^*\xi_\tau\pi(x) = \xi\pi(x).$$

Write $\xi = v|\xi|$ for the polar decomposition of ξ in the standard form $\mathrm{L}^2(M)$. We have $v \in pMq$, $v \neq 0$ and $|\xi| \in \mathrm{L}^2(M)_+$. For every $u \in \mathcal{U}(pPp)$, we have

$$uv\,|\xi| = u\xi = \xi\pi(u) = v|\xi|\pi(u) = v\pi(u)\,\pi(u)^*|\xi|\pi(u).$$

By uniqueness of the polar decomposition, we have $uv = v\pi(u)$ and $|\xi| = \pi(u)^*|\xi|\pi(u)$ for every $u \in \mathcal{U}(pPp)$. It follows that $xv = v\pi(x)$ for every $x \in pPp$. □

Recall that whenever $P \subset M$ is an inclusion of σ-finite von Neumann algebras with expectation such that P is a factor and $M = P \vee (P' \cap M)$, we have $M \cong P \mathbin{\overline{\otimes}} (P' \cap M)$. In that case, we will simply write $M = P \mathbin{\overline{\otimes}} (P' \cap M)$. The next intertwining lemma inside tensor product factors will be crucial in the proof of Theorem 4.3 (see [HI15a, Lemma 4.13] and also [OP03, Proposition 12]).

Lemma 4.2. *Let P_1, Q_1 be any type II_1 factors and P_2, Q_2 any σ-finite type III factors. Put $M = P_1 \mathbin{\overline{\otimes}} P_2$, $N = Q_1 \mathbin{\overline{\otimes}} Q_2$ and assume that $M = N$. If $P_1 \preceq_M Q_1$, then there exist projections $p \in P_1$, $q \in Q_1$ and a nonzero partial isometry $w \in M$ with $ww^* = p$ and $w^*w = q$ such that*

$$qQ_1q = w^*P_1w \mathbin{\overline{\otimes}} B \quad \text{and} \quad w^*P_2w = B \mathbin{\overline{\otimes}} Q_2q.$$

where $B = (w^*P_1w)' \cap qQ_1q$.

Proof. Since $P_1 \preceq_M Q_1$, there exist projections $e \in P_1$ and $f \in Q_1$, a nonzero partial isometry $w \in eMf$ and a unital normal $*$-homomorphism $\pi : eP_1e \to fQ_1f$ such that $xw = w\pi(x)$ for every $x \in eP_1e$. Then we have

$$ww^* \in (eP_1e)' \cap eMe = P_2e \quad \text{and} \quad w^*w \in \pi(eP_1e)' \cap fMf = (\pi(eP_1e)' \cap fQ_1f) \mathbin{\overline{\otimes}} Q_2f.$$

Put $L = \pi(eP_1e)' \cap fQ_1f$ and observe that $\pi(eM_1e)' \cap fMf = L \mathbin{\overline{\otimes}} Q_2f$ is a σ-finite type III von Neumann algebra.

If we denote by $z \otimes 1_{Q_2}f$ the central support in $L \mathbin{\overline{\otimes}} Q_2f$ of the projection $w^*w \in L \mathbin{\overline{\otimes}} Q_2f$, with $z \in \mathcal{Z}(L)$, we have that $w^*w \sim z \otimes 1_{Q_2}f$ in $L \mathbin{\overline{\otimes}} Q_2f$ by [KR97, Corollary 6.3.5]. We may put $q = z$ and assume that $w^*w = q$. Likewise, we have $ww^* \sim 1_{P_2}e$ in P_2e by [KR97, Corollary 6.3.5]. We may put $p = e$ and assume that $ww^* = p$.

Put $B = (w^* P_1 w)' \cap qQ_1 q$. Since $w^* M w = qMq = qQ_1 q \mathbin{\overline{\otimes}} Q_2 q$ and since $w^* P_1 w \subset qQ_1 q$, we obtain

$$w^* P_2 w = (w^* P_1 w)' \cap w^* M w = (w^* P_1 w)' \cap qMq = ((w^* P_1 w)' \cap qQ_1 q) \mathbin{\overline{\otimes}} Q_2 q = B \mathbin{\overline{\otimes}} Q_2 q.$$

Likewise, we obtain $qQ_1 q = w^* P_1 w \mathbin{\overline{\otimes}} B$. ☐

Unique McDuff decomposition. Following [McD69, Co75a], we say that a factor $\mathcal{M}$ is *McDuff* if $\mathcal{M} \cong \mathcal{M} \mathbin{\overline{\otimes}} R$. The following rigidity result is a particular case of the unique McDuff decomposition theorem due to Houdayer–Marrakchi–Verraedt [HMV16, Theorem E]. Let us point out that [HMV16, Theorem E] generalizes Popa's result [Po06, Theorem 5.1] that holds in the type II_1 setting.

Theorem 4.3. *Let M_1 and M_2 be any full type* III *factors with separable predual. If $M_1 \mathbin{\overline{\otimes}} R \cong M_2 \mathbin{\overline{\otimes}} R$, then M_1 and M_2 are isomorphic.*

The proof of Theorem 4.3 is based on Popa's *deformation/rigidity theory*. We will combine Popa's intertwining theory together with the following key rigidity result that allows us to control centralizing sequences in McDuff factors.

Lemma 4.4. *Let M be any full σ-finite factor and N any tracial von Neumann algebra. For every centralizing uniformly bounded net $(x_i)_{i \in I}$ in $M \mathbin{\overline{\otimes}} N$, there exists a centralizing uniformly bounded net $(z_i)_{i \in I}$ in N such that $x_i - z_i \to 0$ $\ast$-strongly.*

Proof. When M is a full semifinite σ-finite factor, the result easily follows from Corollary 3.4. We leave the details to the reader. From now on, we assume that M is a full σ-finite factor of type III.

Fix a faithful normal tracial state τ on N. Since M is a full factor of type III, using Theorem 3.5, we may choose $\kappa > 0$, a faithful state $\varphi \in M_\ast$ and $\xi_1, \dots, \xi_k \in \mathrm{Ball}(M)\xi_\varphi$ such that $J\xi_j = \xi_j$ for every $1 \leq j \leq k$ and such that (3.7) holds. Write $\mathrm{E}_N = \varphi \otimes \mathrm{id}_N : M \mathbin{\overline{\otimes}} N \to N$ for the faithful normal conditional expectation associated with φ on M. We show that

$$\forall x \in M \mathbin{\overline{\otimes}} N, \quad \|x - \mathrm{E}_N(x)\|_{\varphi \otimes \tau}^2 \leq \kappa \sum_{j=1}^{k} \|x(\xi_j \otimes \xi_\tau) - (\xi_j \otimes \xi_\tau)x\|^2. \tag{4.2}$$

By linearity and density, it is enough to prove (4.2) for $x \in M \mathbin{\overline{\otimes}} N$ of the form $x = \sum_{i=1}^m y_i \otimes a_i$ where the family $(a_i \xi_\tau)_{1 \leq i \leq m}$ is chosen to be orthonormal in $\mathrm{L}^2(N)$. In that case, we have

$$\begin{aligned} \|x - \mathrm{E}_N(x)\|_{\varphi \otimes \tau}^2 &= \sum_{i=1}^{m} \|y_i - \varphi(y_i)1\|_\varphi^2 \\ &\leq \kappa \sum_{i=1}^{m} \sum_{j=1}^{k} \|y_i \xi_j - \xi_j y_i\|^2 \\ &= \kappa \sum_{j=1}^{k} \sum_{i=1}^{m} \|y_i \xi_j - \xi_j y_i\|^2 \\ &= \kappa \sum_{j=1}^{k} \|x(\xi_j \otimes \xi_\tau) - (\xi_j \otimes \xi_\tau)x\|^2. \end{aligned}$$

Let now $(x_i)_{i \in I}$ be any centralizing uniformly bounded net in $M \mathbin{\overline{\otimes}} N$. For every $i \in I$, put $z_i = \mathrm{E}_N(x_i) \in N$. Then $(z_i)_{i \in I}$ is a centralizing uniformly bounded net in N and (4.2) shows that $x_i - z_i \to 0$ $\ast$-strongly. ☐

Let us point out that Lemma 4.4 holds for any σ-finite von Neumann algebra N. We refer to [HMV16, Theorem A] for further details.

Proof of Theorem 4.3. Write $R_1 = R = R_2$ so that $\mathcal{M} = M_1 \mathbin{\overline{\otimes}} R_1 = M_2 \mathbin{\overline{\otimes}} R_2$. For every $n \in \mathbf{N}$, write $R_1 = \mathbf{M}_{2^n}(\mathbf{C}) \otimes R_{1,n}$ and observe that $R_{1,n}$ is isomorphic to the hyperfinite type II_1 factor. We start by proving the following claim.

Claim. There exists $n \in \mathbf{N}$ such that $R_{1,n} \preceq_{\mathcal{M}} R_2$.

Indeed, by contradiction, assume that for every $n \in \mathbf{N}$, we have $R_{1,n} \npreceq_{\mathcal{M}} R_2$. Fix a faithful normal conditional expectation $\mathrm{E}_{R_2} : M \to R_2$. By Theorem 4.1, there exists $u_n \in \mathcal{U}(R_{1,n})$ such that $\|\mathrm{E}_{R_2}(u_n)\|_2 \leq \frac{1}{n+1}$. Using Lemma 1.8, $(u_n)_{n\in\mathbf{N}}$ is a uniformly bounded centralizing sequence in M. By Lemma 4.4, there exists a centralizing uniformly bounded sequence $(v_n)_{n\in\mathbf{N}}$ in R_2 such that $u_n - v_n \to 0$ $\ast$-strongly. We then have

$$\lim_n \|v_n\|_2 = \lim_n \|\mathrm{E}_{R_2}(v_n)\|_2 = \lim_n \|\mathrm{E}_{R_2}(v_n - u_n)\|_2 = 0.$$

This implies that $v_n \to 0$ $\ast$-strongly and contradicts the fact that $u_n \in \mathcal{U}(M)$ for every $n \in \mathbf{N}$. This finishes the proof of the claim.

Let $n \in \mathbf{N}$ such that $R_{1,n} \preceq_{\mathcal{M}} R_2$. Using Lemma 4.2, there exist projections $p \in R_{1,n}$, $q \in R_2$ and a nonzero partial isometry $w \in M$ such that $ww^* = p$, $w^*w = q$ and a subfactor $B \subset qR_2q$ such that

$$qR_2q = w^*R_{1,n}w \mathbin{\overline{\otimes}} B \quad \text{and} \quad w^*(M_1 \otimes \mathbf{M}_{2^n}(\mathbf{C}))w = B \mathbin{\overline{\otimes}} M_2q.$$

Since qR_2q is a tracial amenable factor, B is necessarily a tracial amenable factor. Since $w^*(M_1 \otimes \mathbf{M}_{2^n}(\mathbf{C}))w$ is a full factor, B is necessarily a finite type I factor. Thus, M_1 and M_2 are (stably) isomorphic. □

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