STATIONARY ACTIONS OF HIGHER RANK LATTICES ON VON NEUMANN ALGEBRAS

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ABSTRACT. These are the notes of a lecture series given at the Workshop on Operator Algebras held at RIMS, Kyoto University, during January 20-22, 2020. In this lecture series, I discuss a recent joint work with Rémi Boutonnet [BH19] in which we show that for higher rank lattices (e.g. $\mathrm{SL}_3(\mathbf{Z})$), the left regular representation λ is weakly contained in any weakly mixing unitary representation π. This strengthens Margulis' normal subgroup theorem [Ma91], Stuck–Zimmer's stabilizer rigidity result [SZ92] as well as Peterson's character rigidity result [Pe14]. We also show that Uniformly Recurrent Subgroups (URS) of higher rank lattices are finite, answering a question of Glasner–Weiss [GW14]. I will explain the main novelty of our work that consists in proving a structure theorem for stationary actions of higher rank lattices on von Neumann algebras.

CONTENTS

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1. Lecture 1: Introduction and statement of the main results

Introduction. A major achievement in the theory of discrete subgroups of semisimple Lie groups is Margulis' *superrigidity theorem*. The statement is as follows: whenever G is a connected semisimple Lie group with trivial center, no compact factor and real rank at least two, $\Gamma < G$ is an irreducible lattice and H is a simple Lie group with trivial center, any homomorphism $\pi : \Gamma \to H$ such that $\pi(\Gamma)$ is Zariski dense in H and not relatively compact in H extends to a continuous homomorphism $\pi : G \to H$ (see [Ma91, Chapter VII] for more general statements). Connes suggested that there should be a rich analogy between the embedding of a lattice in its ambient Lie group and the embedding of a lattice in its ambient group von Neumann algebra (see [Jo00]).

The first *operator algebraic superrigidity* theorem was obtained by Bekka [Be06] who showed that when $\Gamma = \mathrm{PSL}_n(\mathbf{Z})$ with $n \geq 3$ and M is a type II_1 factor, any homomorphism $\pi : \Gamma \to \mathcal{U}(M)$ such that $\pi(\Gamma)'' = M$ extends to a normal unital $\ast$-isomorphism $\pi : \mathrm{L}(\Gamma) \to M$. Recently, Peterson [Pe14] obtained a far-reaching generalization of Bekka's result by showing that any irreducible lattice $\Gamma < G$ in a property (T) connected semisimple Lie group with trivial center and real rank at least two is operator algebraic superrigid in the above sense (see also [CP13]). Operator algebraic superrigidity can be reformulated as a classification problem for characters on the group.

Recall that a *character* on a countable discrete group Λ is a normalized positive definite function that is conjugation invariant. Thanks to the GNS construction, any character on Λ gives rise to a homomorphism of Λ into the unitary group of a tracial von Neumann algebra. For example, the natural embedding of Λ in its group von Neumann algebra $\mathrm{L}(\Lambda)$ corresponds to the Dirac character δ_e. A rich source of characters comes from ergodic theory of group actions. Indeed, for any probability measure preserving (pmp) action $\Lambda \curvearrowright (X, \nu)$, the map $\varphi : \Lambda \to \mathbf{C} : \gamma \mapsto \nu(\{x \in X \mid \gamma x = x\})$ defines a character on Λ. Then $\varphi = \delta_e$ if and only if the action $\Lambda \curvearrowright (X, \nu)$ is essentially free. The set $\mathrm{Char}(\Lambda)$ of all characters on a countable discrete group Λ is a compact convex set with respect to pointwise convergence. The group Λ is operator algebraic superrigid in the above sense if and only if every extreme point $\varphi \in \mathrm{Char}(\Lambda)$ is either almost periodic (i.e. the corresponding GNS representation is finite dimensional) or $\varphi = \delta_e$. See [Be19, PT13] for other character rigidity results.

Terminology. Before stating our main results, we introduce the following notation that we will use throughout these lectures.

- Let G be any connected simple Lie group with finite center and reak rank at least two. Choose a maximal compact subgroup $K < G$ and a minimal parabolic subgroup $P < G$, so that $G = KP$. Let $\Gamma < G$ be any lattice. We then simply say that $\Gamma < G$ is a *higher rank lattice*.
- We denote by $\nu_P \in \mathrm{Prob}(G/P)$ the unique K-invariant Borel probability measure on the homogeneous space G/P. More generally, if $P \subset Q \subset G$ is a parabolic subgroup, we denote by $\nu_Q \in \mathrm{Prob}(G/Q)$ the unique K-invariant Borel probability measure on the homogeneous space G/Q. Observe that for every parabolic subgroup $P \subset Q \subset G$, the probability measure $\nu_Q \in \mathrm{Prob}(G/Q)$ is G-quasi-invariant.

Example. For every $n \geq 3$, let $G = \mathrm{SL}_n(\mathbf{R})$. Then we may choose $K = \mathrm{SO}_n(\mathbf{R})$, $P < G$ the subgroup of upper triangular matrices and $\Gamma = \mathrm{SL}_n(\mathbf{Z})$.

To simplify the exposition, we choose to state our main results in the setting of *simple* Lie groups. We refer to [BH19] where results hold more generally for *semisimple* Lie groups.

Representation and URS rigidity for higher rank lattices. By [BCH94], when G is a noncompact simple Lie group with trivial center, any lattice $\Gamma < G$ is C^*-*simple* and has the *unique trace property*, that is, the reduced C^*-algebra $\mathrm{C}^*_\lambda(\Gamma)$ is simple and has a unique tracial state τ_Γ (see also [BKKO14] for a new approach). Our first main result provides a far-reaching generalization of this phenomenon to arbitrary weakly mixing representations of higher rank lattices. Recall that a unitary representation π is *weakly mixing* if π does not contain any nonzero finite dimensional subrepresentation.

Theorem A (Representation rigidity [BH19])**.** *Let G be as in the notation and assume moreover that G has trivial center. Let $\Gamma < G$ be any lattice. Then for any weakly mixing representation $\pi : \Gamma \to \mathcal{U}(H_\pi)$, the left regular representation λ is weakly contained in π. Moreover, if we let $\Theta : \mathrm{C}^*_\pi(\Gamma) \to \mathrm{C}^*_\lambda(\Gamma) : \pi(\gamma) \mapsto \lambda(\gamma)$ be the corresponding surjective unital $*$-homomorphism, then*

(i) *$\tau_\Gamma \circ \Theta$ is the unique tracial state on $\mathrm{C}^*_\pi(\Gamma)$.*
(ii) *$\ker(\Theta)$ is the unique proper maximal ideal of $\mathrm{C}^*_\pi(\Gamma)$.*

The next proposition explains why one may regard Theorem A as a strengthening of Margulis' normal subgroup theorem ([Ma91, Chapter IV]), Stuck–Zimmer's stabilizer rigidity result [SZ92] and Peterson's character rigidity result [Pe14].

Proposition 1.1. *Let Λ be any infinite countable discrete group that satisfies the conclusion of Theorem A. Then*

(i) *Λ is* just infinite*, that is, any nontrivial normal subgroup $N < \Lambda$ has finite index.*
(ii) *Λ is* stabilizer rigid*, that is, any properly ergodic pmp action is essentially free.*
(iii) *Λ is* character rigid*, that is, any extreme character is either almost periodic or is δ_e.*

Proof. Let Λ be any infinite countable discrete group that satisfies the conclusion of Theorem A.

(i) Let $N < \Lambda$ be any infinite index normal subgroup. Then the left quasi-regular representation $\pi : \Lambda \to \mathcal{U}(\ell^2(\Lambda/N))$ is weakly mixing. Since Λ satisfies the conclusion of Theorem A, there exists a surjective unital $*$-homomorphism $\Theta : \mathrm{C}^*_\pi(\Lambda) \to \mathrm{C}^*_\lambda(\Lambda)$ such that $\Theta(\pi(\gamma)) = \lambda(\gamma)$ for every $\gamma \in \Lambda$. Since $N < \Lambda$ is normal, for every $\gamma \in N$, we have $\pi(\gamma) = 1$ and so $\lambda(\gamma) = \Theta(\pi(\gamma)) = \Theta(1) = 1$. Since λ is faithful, it follows that $N = \mathbf{1}$.

(ii) Let $\Lambda \curvearrowright (X, \nu)$ be any properly ergodic pmp action. Denote by $\mathcal{R}$ the corresponding orbit equivalence relation and by $\pi : \Lambda \to \mathcal{U}(\mathrm{L}^2(\mathcal{R}))$ the unitary representation defined by $(\pi(\gamma)\eta)(x, y) = \eta(\gamma^{-1}x, y)$ for $\gamma \in \Lambda$, $\eta \in \mathrm{L}^2(\mathcal{R})$, $(x, y) \in \mathcal{R}$. Denote by $\xi = \mathbf{1}_\Delta \in \mathrm{L}^2(\mathcal{R})$ the unit vector corresponding to the diagonal $\Delta \subset \mathcal{R}$ and observe that $\tau = \langle \cdot \xi, \xi\rangle$ is a tracial state on $\mathrm{C}^*_\pi(\Lambda)$. Since $\mathcal{R}$ has infinite orbits almost everywhere, [PT13, Proposition 3.1] implies that π is weakly mixing. Since Λ satisfies the conclusion of Theorem A, we have $\nu(\mathrm{Fix}(\gamma)) = \langle \pi(\gamma)\xi, \xi\rangle = \delta_{\gamma,e}$ for every $\gamma \in \Lambda$. This implies that $\Lambda \curvearrowright (X, \nu)$ is essentially free.

(iii) Let $\varphi \in \mathrm{Char}(\Lambda)$ be any extreme point. Denote by (π, H, ξ) the corresponding GNS triple. Assume that φ is not almost periodic. By extremality, $\pi(\Gamma)''$ is a type II_1 factor and so π is necessarily weakly mixing. Since Λ satisfies the conclusion of Theorem A, we have $\varphi(\gamma) = \langle \pi(\gamma)\xi, \xi\rangle = \delta_{\gamma,e}$ for every $\gamma \in \Lambda$. This implies that $\varphi = \delta_e$. □

Denote by $\mathrm{Sub}(\Gamma)$ the compact metrizable space of all subgroups of Γ endowed with the Chabauty topology. Define the conjugation action $\Gamma \curvearrowright \mathrm{Sub}(\Gamma)$ by $\gamma \cdot \Lambda = \gamma\Lambda\gamma^{-1}$ for $\gamma \in \Gamma$ and $\Lambda \in \mathrm{Sub}(\Gamma)$. In the measurable setting, following [AGV12], an *invariant random subgroup* (IRS for short) is a conjugation invariant Borel probability measure $\nu \in \mathrm{Prob}(\mathrm{Sub}(\Gamma))$. By Stuck–Zimmer's stabilizer rigidity result [SZ92, Corollary 4.4], any ergodic IRS of any lattice $\Gamma < G$, where G is as in the notation and with trivial center, is finite. In the topological setting, following [GW14], a *uniformly recurrent subgroup* (URS for short) is a nonempty closed minimal Γ-invariant subset $X \subset \mathrm{Sub}(\Gamma)$. The study of URSs has received a lot of attention recently due to its connections with IRSs ([7s12, Ge14]) and C^*-simplicity ([Ke15, LBMB16]). Our second main result provides a topological analogue of Stuck–Zimmer's stabilizer rigidity result [SZ92] and answers positively a question raised by Glasner–Weiss (see [GW14, Problem 5.4]).

Theorem B (URS rigidity [BH19])**.** *Let G be as in the notation and assume moreover that G has trivial center. Let $\Gamma < G$ be any lattice. For any minimal action $\Gamma \curvearrowright X$ on a compact metrizable space, either X is finite or the action $\Gamma \curvearrowright X$ is topologically free. In particular, any* URS *of Γ is finite.*

Classification of stationary characters of higher rank lattices. Let us introduce some terminology regarding *Poisson boundaries*. Let H be any locally compact second countable (lcsc) group and $\mu \in \mathrm{Prob}(H)$ any *admissible* Borel probability measure, that is, μ is equivalent to the Haar measure. Let X be any standard Borel space, $\nu \in \mathrm{Prob}(X)$ any Borel probability measure and $H \curvearrowright X$ any Borel action. We say that ν is *μ-stationary* if $\mu * \nu = \nu$. We then say that (X, ν) is a (H, μ)-space. In that case, the measure $\nu \in \mathrm{Prob}(X)$ is necessarily H-quasi-invariant (see e.g. [NZ97, Lemma 1.1]). Following [Fu62a, BS04], we say that a bounded function $f : H \to \mathbf{C}$ is (right) *μ-harmonic* if $f(g) = \int_H f(gh)\, \mathrm{d}\mu(h)$ for every $g \in H$. Any bounded μ-harmonic function is necessarily continuous. Denote by $\mathrm{Har}^\infty(H, \mu)$ the (left) H-space of all bounded μ-harmonic functions on H. We denote by (B, ν_B) the (H, μ)-*Poisson boundary* (see [Fu62b, Fu00]). The Poisson boundary (B, ν_B) is an ergodic (H, μ)-space that enjoys remarkable structural properties. The first result we need regarding Poisson boundaries is the following fundamental fact.

Lemma 1.2 ([BS04, Theorem 2.11])**.** *The linear mapping*

$$\widehat{\cdot} : \mathrm{L}^\infty(B, \nu_B) \to \mathrm{Har}^\infty(H, \mu) : f \mapsto \widehat{f}$$

where $\widehat{f}(g) = \int_B f(gw)\, \mathrm{d}\nu_B(w)$ for every $g \in H$, is H-equivariant, isometric and bijective.

The second result we need regarding Poisson boundaries ensures the existence and uniqueness of boundary maps. Whenever $\mathcal{A}$ is unital H-C*-algebra and $\phi \in \mathcal{S}(\mathcal{A})$ is a state, we say that ϕ is μ-stationary if $\mu * \phi = \phi$. By Markov–Kalutani's lemma, there always exists a μ-stationary state $\phi \in \mathcal{S}(\mathcal{A})$.

Lemma 1.3 ([BH19, Theorem 2.5])**.** *Let $\mathcal{A}$ be any separable unital H-*C**-algebra and $\phi \in \mathcal{S}(\mathcal{A})$ any μ-stationary state. Then there exists an essentially unique H-equivariant measurable boundary map $\beta_\phi : B \to \mathcal{S}(\mathcal{A}) : w \mapsto \phi_w$ that satisfies*

$$\phi = \int_B \phi_w \, \mathrm{d}\nu_B(w).$$

The following concept is central in these lectures.

Definition 1.4 (Furstenberg measure)**.** Let G be as in the notation and $\Gamma < G$ any lattice. We say that a probability measure $\mu_0 \in \mathrm{Prob}(\Gamma)$ is *Furstenberg* if the following three conditions are satisfied:

(i) The support of μ_0 is equal to Γ;
(ii) $\mu_0 * \nu_P = \nu_P$, that is, ν_P is μ_0-stationary;
(iii) The space $(G/P, \nu_P)$ is the (Γ, μ_0)-Poisson boundary.

By a result of Furstenberg [Fu67, Theorem 3] (see also [Fu00, Theorem 2.21]), there always exists a Furstenberg measure $\mu_0 \in \mathrm{Prob}(\Gamma)$. Moreover, for every parabolic subgroup $P \subset Q \subset G$, $\nu_Q \in \mathrm{Prob}(G/Q)$ is the unique μ_0-stationary Borel probability measure on the homogeneous space G/Q (see [Ma91, Corollary VI.3.9]).

Let Λ be any countable discrete group and $\mu \in \mathrm{Prob}(\Lambda)$ any probability measure. We say that a normalized positive definite function $\varphi : \Lambda \to \mathbf{C}$ is a *μ-character* if $\sum_{\gamma \in \Lambda} \mu(\gamma)\, \varphi(\gamma^{-1} g \gamma) = \varphi(g)$ for all $g \in \Lambda$. Any character on Λ is obviously a μ-character. Conversely, our next result shows that when Γ is a higher rank lattice and $\mu_0 \in \mathrm{Prob}(\Gamma)$ is a Furstenberg measure, any μ_0-character is a genuine character. We moreover obtain a new proof of Peterson's *character rigidity* result [Pe14].

Theorem C (Classification of μ_0-characters [BH19])**.** *Let G be as in the notation and assume moreover that G has trivial center. Let $\Gamma < G$ be any lattice and $\mu_0 \in \mathrm{Prob}(\Gamma)$ any Furstenberg measure. The following assertions hold:*

(i) *Any μ_0-character φ on Γ is* conjugation invariant*, that is, φ is a genuine character.*
(ii) *Any extreme point $\varphi \in \mathrm{Char}(\Gamma)$ is either almost periodic or $\varphi = \delta_e$.*

Let us explain how we can deduce Theorem A from Theorem C.

Proof of Theorem A using Theorem C. Assume that G has trivial center. Let $\Gamma < G$ be any lattice. Since Γ has property (T), Γ has only countably many finite dimensional unitary representations up to unitary conjugacy, that we denote by π_n, $n \geq 1$ (see [Wa74, Theorem 2.1(iv)]). Denote by $\pi_0 = \lambda$ the left regular representation. Theorem C (ii) implies that Γ has countably many extreme points in the space of characters: those corresponding to the finite dimensional unitary representations (π_n, H_n), denoted by τ_n, $n \geq 1$; and the one corresponding to the left regular representation $\pi_0 = \lambda$, namely the Dirac map at the identity, denoted by $\tau_0 = \delta_e$.

Let π be any weakly mixing unitary representation. Set $A = \mathrm{C}^*_\pi(\Gamma)$ and denote by $\sigma : \Gamma \curvearrowright A$ the conjugation action defined by $\sigma_\gamma = \mathrm{Ad}(\pi(\gamma))$ for every $\gamma \in \Gamma$. By Markov–Kakutani's lemma, there exists a μ_0-stationary state $\phi \in \mathcal{S}(A)$. Since $\varphi = \phi \circ \pi$ is a μ_0-character, it is in fact a genuine character by Theorem C (i). This means that ϕ is a tracial state. Thus, A has at least one tracial state. We now prove that the left regular representation λ is weakly contained in π and that A has a unique tracial state.

Let $\phi \in \mathcal{S}(A)$ be any tracial state and denote by $\varphi = \phi \circ \pi$ the corresponding character on Γ. By the first paragraph, we may find a sequence $(\alpha_n)_{n\in\mathbf{N}}$ of nonnegative real numbers such that $1 = \sum_{n\in\mathbf{N}} \alpha_n$ and $\varphi = \sum_{n\in\mathbf{N}} \alpha_n \tau_n$.

Claim 1.5. For every $n \in \mathbf{N}$ such that $\alpha_n \neq 0$, we have that π_n is weakly contained in π.

Indeed, we view π_n and π as representations of the full C*-algebra $\mathrm{C}^*(\Gamma)$. Regard $\varphi = \phi \circ \pi$ as a state on $\mathrm{C}^*(\Gamma)$. We denote by ϕ_n the canonical faithful normal tracial state on the finite factor $\pi_n(\Gamma)''$ so that $\phi_n \circ \pi_n = \tau_n$. Note that ϕ_n is implemented by a cyclic vector $\xi_n \in H_n$. By uniqueness of the GNS representation, we have that (π_n, H_n, ξ_n) is the GNS triple associated with τ_n. For all $a, b \in \mathrm{C}^*(\Gamma)$, we have

$$\|\pi_n(b)\, \pi_n(a)\xi_n\| = \|ba\|_{2,\tau_n} \leq \frac{1}{\sqrt{\alpha_n}} \|ba\|_{2,\phi\circ\pi} \leq \frac{1}{\sqrt{\alpha_n}} \|\pi(b)\| \cdot \|\pi(a)\|.$$

If $\pi(b) = 0$, then $\pi_n(b) = 0$. This proves our claim.

Using Claim 1.5, we obtain that $\alpha_n = 0$ for every $n \geq 1$. Then $\varphi = \tau_0$ and Claim 1.5 implies that $\lambda = \pi_0$ is weakly contained in π. Denote by $\Theta : A \to \mathrm{C}^*_\lambda(\Gamma) : \pi(\gamma) \mapsto \lambda(\gamma)$ the corresponding surjective unital $*$-homomorphism. Then we have $\phi = \tau_0 \circ \Theta$. This shows that $A = \mathrm{C}^*_\pi(\Gamma)$ has a unique tracial state.

Finally, we show that $\ker(\Theta)$ is the unique proper maximal ideal of $A = \mathrm{C}^*_\pi(\Gamma)$. Assume that $I \subset A$ is a proper ideal and consider the quotient map $\alpha : A \to A/I$. Then the unitary representation $\rho : \Gamma \to \mathcal{U}(A/I) : \gamma \mapsto \alpha(\pi(\gamma))$ is weakly contained in π and hence is weakly mixing (because π is weakly mixing and Γ has property (T)). By the first part of the proof, we know that λ is weakly contained in ρ, which in turn implies that the map $\beta : A/I \to \mathrm{C}^*_\lambda(\Gamma) : \alpha(\pi(\gamma)) \mapsto \lambda(\gamma)$ is a well-defined surjective unital $*$-homomorphism. By construction, we have $\Theta = \beta \circ \alpha$. This shows that $I = \ker(\alpha) \subset \ker(\Theta)$. □

Our main rigidity results apply to C*-algebras, unitary representations and topological dynamics associated with higher rank lattices. As we will explain in the next lecture, all these rigidity results are deduced from a general structure theorem for stationary actions of higher rank lattices on *von Neumann algebras*.

2. Lecture 2: The structure theorem and its consequences

The structure theorem for stationary actions of higher rank lattices. Let H be any lcsc group and $\mu \in \mathrm{Prob}(H)$ any admissible Borel probability measure. Let $\mathcal{M}$ be any H-von Neumann algebra and $\varphi \in \mathcal{M}_*$ any normal state. We say that the action $\sigma : H \curvearrowright \mathcal{M}$ is *ergodic* if the fixed-point subalgebra $\mathcal{M}^H = \{x \in \mathcal{M} \mid \sigma_g(x) = x, \forall g \in H\}$ satisfies $\mathcal{M}^H = \mathbf{C}1$. As in the previous lecture, we say that the state $\varphi \in \mathcal{M}_*$ is *μ-stationary* if $\mu * \varphi = \varphi$. If the action $\sigma : H \curvearrowright \mathcal{M}$ is ergodic and the state $\varphi \in \mathcal{M}_*$ is μ-stationary, we say that $(\mathcal{M}, \varphi)$ is an ergodic (H, μ)-von Neumann algebra.

The main novelty of our work is the following structure theorem for stationary actions of higher rank lattices on von Neumann algebras.

Theorem D (Structure of μ_0-stationary actions [BH19])**.** *Let G be as in the notation. Let $\Gamma < G$ be any lattice and $\mu_0 \in \mathrm{Prob}(\Gamma)$ any Furstenberg measure. Let (M, ϕ) be any ergodic (Γ, μ_0)-von Neumann algebra. Then the following dichotomy holds:*

- *Either ϕ is Γ-invariant.*
- *Or there exist a proper parabolic subgroup $P \subset Q \subsetneq G$ and a Γ-equivariant normal unital $*$-embedding $\theta : \mathrm{L}^\infty(G/Q, \nu_Q) \to M$ such that $\phi \circ \theta = \nu_Q$.*

We point out that this result is even new in the case when M is abelian. Let us first explain how we can deduce Theorem B from Theorem D.

Proof of Theorem B using Theorem D. Assume that G has trivial center. Let $\Gamma < G$ be any lattice. Choose a Furstenberg measure $\mu_0 \in \mathrm{Prob}(\Gamma)$. Let $\Gamma \curvearrowright X$ be any minimal action. Choose an extreme point ν in the compact convex set of all μ_0-stationary Borel probability measures on the compact metrizable space X. Then the action $\Gamma \curvearrowright (X, \nu)$ is ergodic. Moreover, since the action $\Gamma \curvearrowright X$ is minimal, we have $\mathrm{supp}(\nu) = X$. Assume that X is not finite. By minimality and μ_0-stationary, the probability space (X, ν) is necessarily diffuse. Moreover, since $\Gamma \curvearrowright X$ is minimal, in order to show that $\Gamma \curvearrowright X$ is topologically free, it suffices to show that $\Gamma \curvearrowright (X, \nu)$ is essentially free. There are two cases to consider:

- Assume that ν is Γ-invariant. Then the action $\Gamma \curvearrowright (X, \nu)$ is essentially free by Stuck–Zimmer's stabilizer rigidity result [SZ92, Corollary 4.4].
- Assume that ν is not Γ-invariant. By Theorem D, there exist a proper parabolic subgroup $P \subset Q \subsetneq G$ and a Γ-equivariant measurable factor map $(X, \nu) \to (G/Q, \nu_Q)$. Since $\Gamma \curvearrowright (G/Q, \nu_Q)$ is essentially free by Lemma 2.1 below, it follows that $\Gamma \curvearrowright (X, \nu)$ is essentially free.

Let now $X \subset \mathrm{Sub}(\Gamma)$ be any URS. By definition of the URS and since $\Gamma \neq \{e\}$, for every $x = \Lambda \in X$, we have $\mathrm{Stab}_\Gamma(x) = \mathrm{N}_\Gamma(\Lambda) \neq \{e\}$. This implies that $\Gamma \curvearrowright X$ is not topologically free. Therefore, X is finite. □

Classification of stationary characters using the structure theorem. The goal of this lecture is to prove Theorem C using Theorem D. We first need to prove a few preliminary results. The following result is essentially contained in [Oz16, Remark 13].

Lemma 2.1 ([BH19, Lemma 6.2])**.** *Let G be as in the notation and assume moreover that G has trivial center. Let $\Gamma < G$ be any lattice and $H < G$ any proper closed subgroup. Denote by $\nu_{G/H} \in \mathrm{Prob}(G/H)$ a G-quasi-invariant Borel probability measure. Then $\Gamma \curvearrowright (G/H, \nu_{G/H})$ is essentially free.*

Proof. Set $N = \bigcap_{g \in G} gHg^{-1}$ and note that N is proper closed normal subgroup of G. Since G is a simple Lie group with trivial center, it follows that $N = \mathbf{1}$.

Let $\gamma \in \Gamma$ be any element such that $\nu_{G/H}(\{gH \in G/H \mid \gamma gH = gH\}) > 0$. Let $d = \dim(G)$. By [Oz16, Remark 13], for almost every $(g_1, \dots, g_{d+1}) \in G^{d+1}$, we have $\bigcap_{i=1}^{d+1} g_i H g_i^{-1} = N$. This implies that $\gamma \in \Gamma \cap N$ and so $\gamma = e$. □

The next lemma is inspired by Hartman–Kalantar [HK17, Example 4.11] and Haagerup [Ha15, Lemma 3.1].

Lemma 2.2 ([BH19, Lemma 6.3])**.** *Let G be as in the notation and assume moreover that G has trivial center. Let $\Gamma < G$ be any lattice and $\mu_0 \in \operatorname{Prob}(\Gamma)$ any Furstenberg measure. Let B be any unital C^*-algebra and $\pi : \Gamma \to \mathcal{U}(B)$ any unitary representation. Consider the conjugation action $\sigma : \Gamma \curvearrowright B$ defined by $\sigma_\gamma = \operatorname{Ad}(\pi(\gamma))$ for every $\gamma \in \Gamma$.*

Assume that there exist a proper parabolic subgroup $P \subset Q \subsetneq G$ and a Γ-equivariant unital $*$-homomorphism $\theta : \mathrm{C}(G/Q) \to B$. Let $\phi \in \mathcal{S}(B)$ be any μ_0-stationary state. Then we have $\phi \circ \pi = \delta_e$.

Proof. Let $\phi \in \mathcal{S}(B)$ be any μ_0-stationary state. Denote by $A \subset B$ the *separable* unital C^*-subalgebra generated by $\pi(\Gamma)$ and $\theta(\mathrm{C}(G/Q))$ and observe that $A \subset B$ is globally Γ-invariant under the action σ. Since A is separable and since $\phi|_A \in \mathcal{S}(A)$ is μ_0-stationary, Lemma 1.3 shows that there exists an essentially unique Γ-equivariant measurable boundary map $\beta_\phi : G/P \to \mathcal{S}(A) : w \mapsto \phi_w$ that satisfies $\phi = \int_{G/P} \phi_w \, \mathrm{d}\nu_P(w)$. Denote by $p_Q : G/P \to G/Q$ the factor map that moreover satisfies $(p_Q)_*\nu_P = \nu_Q$. Since the state $\phi \circ \theta$ is μ_0-stationary on $\mathrm{C}(G/Q)$ and since ν_Q is the only μ_0-stationary state on $\mathrm{C}(G/Q)$ (see [Ma91, Corollary VI.3.9]), we infer that

$$\int_{G/P} \phi_w \circ \theta \, \mathrm{d}\nu_P(w) = \phi \circ \theta = \nu_Q = \int_{G/P} \delta_{p_Q(w)} \, \mathrm{d}\nu_P(w).$$

By Lemma 1.3, the Γ-equivariant measurable maps $G/P \to \mathcal{S}(\mathrm{C}(G/Q)) : w \mapsto \phi_w \circ \theta$ and $G/P \to \mathcal{S}(\mathrm{C}(G/Q)) : w \mapsto \delta_{p_Q(w)}$ coincide almost everywhere. Thus there exists a conull measurable subset $\Omega_1 \subset G/P$ such that $\phi_w \circ \theta = \delta_{p_Q(w)}$ for every $w \in \Omega_1$. Since $\delta_{p_Q(w)} \in \mathcal{S}(\mathrm{C}(G/Q))$ is multiplicative, we infer that $\theta(\mathrm{C}(G/Q))$ lies in the multiplicative domain of ϕ_w for every $w \in \Omega_1$.

Fix $\gamma \in \Gamma \setminus \{e\}$. Since $\nu_Q(\{y \in G/Q \mid \gamma y = y\}) = 0$ by Lemma 2.1, we may find a conull measurable subset $\Omega_2 \subset G/P$ such that $\gamma p_Q(w) \neq p_Q(w)$ for all $w \in \Omega_2$. Fix $w \in \Omega_1 \cap \Omega_2$, set $y = p_Q(w) \in G/Q$ and choose a continuous function $f \in \mathrm{C}(G/Q)$ such that $f(y) = 1$ and $f(\gamma y) = 0$. We compute

$$\phi_w(\pi(\gamma)) = f(y)\, \phi_w(\pi(\gamma)) = \phi_w(\theta(f)\, \pi(\gamma)) = \phi_w(\pi(\gamma)\, \theta(\sigma_\gamma^{-1}(f))) = \phi_w(\pi(\gamma))\, f(\gamma y) = 0.$$

By integrating with respect to $w \in \Omega_1 \cap \Omega_2$, we obtain $\phi(\pi(\gamma)) = 0$. □

We are now ready to prove Theorem C using Theorem D.

Proof of Theorem C using Theorem D. Let G be as in the notation and assume moreover that G has trivial center. Let $\Gamma < G$ be any lattice and $\mu_0 \in \operatorname{Prob}(\Gamma)$ any Furstenberg measure.

(i) Observe that the set $\operatorname{Char}_{\mu_0}(\Gamma)$ of all μ_0-characters on Γ is a nonempty compact convex subset of the space $\ell^\infty(\Gamma)$ endowed with the weak* topology. Using Krein–Milman's theorem, in order to prove Theorem C, it suffices to show that any extreme point in $\operatorname{Char}_{\mu_0}(\Gamma)$ is conjugation invariant.

Let $\varphi \in \operatorname{Char}_{\mu_0}(\Gamma)$ be any extreme point. Denote by (π, H, ξ) the GNS triple corresponding to φ and set $M = \pi(\Gamma)''$. Recall that $\varphi(\gamma) = \langle \pi(\gamma)\xi, \xi \rangle$ for every $\gamma \in \Gamma$. We denote by ϕ the normal state $\langle \cdot\, \xi, \xi \rangle$ on M and observe that $\varphi = \phi \circ \pi$. Denote by $\sigma : \Gamma \curvearrowright M$ the conjugation action defined by $\sigma_\gamma = \operatorname{Ad}(\pi(\gamma))$ for every $\gamma \in \Gamma$. Then $\phi \in M_*$ is a normal μ_0-stationary state. Observe that $\varphi \in \operatorname{Char}_{\mu_0}(\Gamma)$ is conjugation invariant if and only if $\phi \in M_*$ is Γ-invariant.

First, we prove that $\phi \in M_*$ is faithful. Indeed, let $x \in M$ be any element such that $\phi(x^*x) = 0$. Since $\mu_0 * \phi = \phi$, we obtain

$$\sum_{\gamma \in \Gamma} \mu_0(\gamma)\, \|x\, \pi(\gamma)\xi\|^2 = \sum_{\gamma \in \Gamma} \mu_0(\gamma)\, \phi(\pi(\gamma)^*\, x^*x\, \pi(\gamma)) = (\mu_0 * \phi)(x^*x) = \phi(x^*x) = 0.$$

This implies that $x\, \pi(\gamma)\xi = 0$ for all $\gamma \in \operatorname{supp}(\mu_0) = \Gamma$. Since ξ is $\pi(\Gamma)$-cyclic, we conclude that $x\eta = 0$ for all $\eta \in H$ and so $x = 0$. Thus, $\phi \in M_*$ is a faithful normal state.

Next, we prove that the action $\Gamma \curvearrowright M$ is ergodic. We prove the contrapositive statement. Assume that $M^\Gamma = \mathcal{Z}(M)$ is nontrivial. Then there is a nontrivial projection $p \in \mathcal{Z}(M) \setminus \{0, 1\}$. Since ϕ is faithful, $\phi(p) \notin \{0, 1\}$. Define the μ_0-stationary normal states $\phi_1 = \frac{1}{\phi(p)}\phi(\,\cdot\, p) \in M_*$ and $\phi_2 = \frac{1}{\phi(1-p)}\phi(\,\cdot\,(1-p)) \in M_*$. We have $\phi = \phi(p)\,\phi_1 + \phi(1-p)\,\phi_2$. We have $\phi_1(p) = 1$ and $\phi_2(p) = 0$ so that $\phi_1 \neq \phi_2$. Define the μ_0-characters $\varphi_1 = \phi_1 \circ \pi \in \mathrm{Char}_{\mu_0}(\Gamma)$ and $\varphi_2 = \phi_2 \circ \pi \in \mathrm{Char}_{\mu_0}(\Gamma)$. We have $\varphi = \phi(p)\,\varphi_1 + \phi(1-p)\,\varphi_2$. Since ϕ_1 and ϕ_2 are normal, since $\phi_1 \neq \phi_2$ and since the linear span of $\pi(\Gamma)$ is ultraweakly dense in M, there exists $\gamma \in \Gamma$ such that $\varphi_1(\gamma) = \phi_1(\pi(\gamma)) \neq \phi_2(\pi(\gamma)) = \varphi_2(\gamma)$. This implies that $\varphi_1 \neq \varphi_2$ and hence φ is not an extreme point in $\mathrm{Char}_{\mu_0}(\Gamma)$. This shows that the action $\Gamma \curvearrowright M$ is ergodic.

Then the action $\Gamma \curvearrowright M$ is ergodic and $\phi \in M_*$ is a μ_0-stationary faithful normal state. Assume by contradiction that ϕ is not Γ-invariant. By Theorem D, there exist a proper parabolic subgroup $P \subset Q \subsetneq G$ and a Γ-equivariant unital $*$-homomorphism $\theta : \mathrm{C}(G/Q) \to M$. By Lemma 2.2, we obtain that $\varphi = \phi \circ \pi = \delta_e$ is conjugation invariant. This further implies that ϕ is Γ-invariant, contradicting our assumption.

(ii) Let $\varphi \in \mathrm{Char}(\Gamma)$ be any extreme point. Denote by (π, H, ξ) the GNS triple corresponding to φ. The von Neumann algebra $M = \pi(\Gamma)''$ is a finite factor and the vector state $\phi = \langle \cdot\, \xi, \xi\rangle \in M_*$ is the canonical faithful normal trace. Denote by $J : H \to H : x\xi \mapsto x^*\xi$ the canonical anti-unitary. We have $JMJ = M' \cap \mathbf{B}(H)$. The Hilbert space H is naturally endowed with an M-M-bimodule structure given by $x\eta y = xJy^*J\eta$ for all $x, y \in M$ and all $\eta \in H$.

Following [Pe14], define $\mathcal{B} = \left(\mathrm{L}^\infty(G/P) \mathbin{\overline{\otimes}} \mathbf{B}(H)\right)^\Gamma$ the fixed-point von Neumann algebra with respect to the Γ-action $\alpha : \Gamma \curvearrowright \mathrm{L}^\infty(G/P, \nu_P) \mathbin{\overline{\otimes}} \mathbf{B}(H)$ defined by $\alpha_\gamma = \sigma_\gamma \otimes \mathrm{Ad}(J\pi(\gamma)J)$ for every $\gamma \in \Gamma$. Here, $\sigma : \Gamma \curvearrowright \mathrm{L}^\infty(G/P, \nu_P)$ is the natural translation action. Alternatively, we can view $\mathcal{B}$ as the von Neumann algebra of all essentially bounded measurable functions $f : G/P \to \mathbf{B}(H)$, modulo equality ν_P-almost everywhere, that satisfy $f(\gamma w) = \mathrm{Ad}(J\pi(\gamma)J)(f(w))$ for every $\gamma \in \Gamma$ and ν_P-almost every $w \in G/P$. Note that $\mathbf{C}1 \mathbin{\overline{\otimes}} M \subset \mathcal{B}$ corresponds to the von Neumann subalgebra of all essentially constant measurable functions $f : G/P \to M$. Since P is amenable since $\mathrm{L}^\infty(G/\Gamma) \mathbin{\overline{\otimes}} \mathbf{B}(H)$ is amenable and since $\mathcal{B} \cong (\mathrm{L}^\infty(G/\Gamma) \mathbin{\overline{\otimes}} \mathbf{B}(H))^P$, it follows that $\mathcal{B}$ is amenable (see [CP13, Section 2]).

Define the conjugation action $\beta : \Gamma \curvearrowright \mathrm{L}^\infty(G/P) \mathbin{\overline{\otimes}} \mathbf{B}(H)$ by $\beta_\gamma = \mathrm{Ad}(1 \otimes \pi(\gamma))$ for every $\gamma \in \Gamma$. Then β commutes with α and so $\mathcal{B}$ is globally Γ-invariant under the action β.

Claim 2.3 ([BH19, Lemma 6.4])**.** The following assertions hold true:

(i) The action $\beta : \Gamma \curvearrowright \mathcal{B}$ is ergodic.
(ii) The normal state $\Phi : \mathcal{B} \to \mathbf{C} : f \mapsto \int_{G/P} \langle f(w)\xi, \xi\rangle \, \mathrm{d}\nu_P(w)$ is μ_0-stationary.

Proof of Claim 2.3. (i) Denote by $\mathcal{B}^{\beta(\Gamma)} = \{f \in \mathcal{B} \mid \beta_\gamma(f) = f, \forall \gamma \in \Gamma\}$ the fixed-point von Neumann subalgebra of $\mathcal{B}$ with respect to the action β. Since $JMJ = M' \cap \mathbf{B}(H)$, by construction, $\mathcal{B}^{\beta(\Gamma)}$ is the von Neumann algebra of all essentially bounded measurable functions $f : G/P \to JMJ$, modulo equality ν_P-almost everywhere, that satisfy $f(\gamma w) = \mathrm{Ad}(J\pi(\gamma)J)(f(w))$ for every $\gamma \in \Gamma$ and ν_P-almost every $w \in G/P$. Since JMJ is a finite von Neumann algebra with separable predual, we may view it as a separable metric space with respect to the distance $d : JMJ \times JMJ \to \mathbf{R}_{\geq 0}$ defined by $d(JxJ, JyJ) = \phi((y-x)^*(y-x))^{1/2}$ for all $x, y \in M$. Moreover, the action $\mathrm{Ad}(J\pi(\,\cdot\,)J) : \Gamma \curvearrowright (JMJ, d)$ is isometric since ϕ is a (faithful normal) trace on M. Then [BF14, Theorem 2.5] implies that $\mathcal{B}^{\beta(\Gamma)} \subset \mathbf{C}1$ (we also used that M is a factor).

(ii) For every $f \in \mathcal{B}$, we have

$$\begin{aligned}\sum_{\gamma \in \Gamma} \mu_0(\gamma)\, \Phi(\beta_\gamma^{-1}(f)) &= \sum_{\gamma \in \Gamma} \mu_0(\gamma) \int_{G/P} \langle f(w)\,\pi(\gamma)\xi, \pi(\gamma)\xi\rangle \, \mathrm{d}\nu_P(w) \\ &= \sum_{\gamma \in \Gamma} \mu_0(\gamma) \int_{G/P} \langle f(w)\, J\pi(\gamma)^* J\xi, J\pi(\gamma)^* J\xi\rangle \, \mathrm{d}\nu_P(w)\end{aligned}$$

$$\begin{aligned} &= \sum_{\gamma \in \Gamma} \mu_0(\gamma) \int_{G/P} \langle f(\gamma w)\xi, \xi\rangle \, \mathrm{d}\nu_P(w) \quad (\text{since } f \in \mathcal{B}) \\ &= \int_{G/P} \langle f(w)\xi, \xi\rangle \, \mathrm{d}\nu_P(w) \quad (\text{since } \nu_P \text{ is } \mu_0\text{-stationary}) \\ &= \Phi(f). \end{aligned}$$

Thus, Φ is a μ_0-stationary state. □

By Claim 2.3 and Theorem D, the following dichotomy holds:

(1) Either Φ is Γ-invariant with respect to β.
(2) Or there exist a proper parabolic subgroup $P \subset Q \subsetneq G$ and a Γ-equivariant map $\Theta : \mathrm{C}(G/Q) \to \mathcal{B}$.

If (2) holds, Lemma 2.2 shows that $\Phi \circ (1 \otimes \pi) = \delta_e$ and so $\phi \circ \pi = \delta_e$. If (1) holds, we show that that M is a finite dimensional factor and hence φ is almost periodic. Since Γ has property (T), M has property (T) in the sense of [CJ83] so it suffices to prove that M is amenable. Indeed, any tracial factor that simultaneously has property (T) and that is amenable is necessarily finite dimensional. As we saw, $\mathcal{B}$ is amenable, so we only need to verify that $\mathbf{C}1 \mathbin{\overline{\otimes}} M = \mathcal{B}$.

For every $f \in \mathcal{B}$ and every $\gamma \in \Gamma$, we have

$$\begin{aligned} \Phi(\beta_\gamma^{-1}(f)) &= \int_{G/P} \langle f(w)\, \pi(\gamma)\xi, \pi(\gamma)\xi\rangle \, \mathrm{d}\nu_P(w) \\ &= \int_{G/P} \langle f(w)\, J\pi(\gamma)^* J\xi, J\pi(\gamma)^* J\xi\rangle \, \mathrm{d}\nu_P(w) \\ &= \int_{G/P} \langle f(\gamma w)\xi, \xi\rangle \, \mathrm{d}\nu_P(w). \end{aligned}$$

Since this quantity does not depend on $\gamma \in \Gamma$, the bounded μ_0-harmonic function $\Gamma \to \mathbf{C} : \gamma \mapsto \int_{G/P} \langle f(\gamma w)\xi, \xi\rangle \, \mathrm{d}\nu_P(w)$ is constant. Since $(G/P, \nu_P)$ is the (Γ, μ_0)-Poisson boundary, Lemma 1.2 shows that the bounded measurable function $G/P \to \mathbf{C} : w \mapsto \langle f(w)\xi, \xi\rangle$ is ν_P-almost everywhere constant. Since $\mathbf{C}1 \mathbin{\overline{\otimes}} M \subset \mathcal{B}$, we deduce that for all $f \in \mathcal{B}$ and all $a, b \in M$, $(1 \otimes b^*) f (1 \otimes a) \in \mathcal{B}$ and so the measurable function $G/P \to \mathbf{C} : w \mapsto \langle f(w)\, a\xi, b\xi\rangle$ is essentially constant. By separability of H and density of $M\xi$ in H, we conclude that f is essentially constant. Since $f(\gamma w) = \mathrm{Ad}(J\pi(\gamma)J)(f(w))$ for every $\gamma \in \Gamma$ and ν_P-almost every $w \in G/P$, we conclude that the unique essential value of f commutes with JMJ and so lies in M. This shows that $f \in \mathbf{C}1 \mathbin{\overline{\otimes}} M$. Thus, we have $\mathbf{C}1 \mathbin{\overline{\otimes}} M = \mathcal{B}$. □

Stationary induction. Let G be as in the notation and $\Gamma < G$ any lattice. Let M be any Γ-von Neumann algebra. Define the *induced* von Neumann algebra $\mathrm{Ind}_\Gamma^G(M) = (\mathrm{L}^\infty(G) \mathbin{\overline{\otimes}} M)^\Gamma$ where the action $\Gamma \curvearrowright \mathrm{L}^\infty(G) \mathbin{\overline{\otimes}} M$ is given by $\rho_\gamma \otimes \sigma_\gamma$ for $\gamma \in \Gamma$. We endow $\mathrm{Ind}_\Gamma^G(M)$ with the *induced* G-action given by $(\lambda_g \otimes \mathrm{id}_M)$ for $g \in G$. Then $\mathcal{M} = \mathrm{Ind}_\Gamma^G(M)$ is naturally a G-von Neumann algebra.

Denote by $m_{G/\Gamma} \in \mathrm{Prob}(G/\Gamma)$ the unique G-invariant Borel probability measure. If $\phi \in M_*$ is a Γ-invariant normal state on M, then $\varphi = m_{G/\Gamma} \otimes \phi \in \mathcal{M}_*$ is a G-invariant normal state on $\mathcal{M}$. Our next result shows that under particular assumptions, starting from a stationary normal state on the Γ-von Neumann algebra M, one can naturally define a stationary normal state on the induced G-von Neumann algebra $\mathcal{M}$.

Theorem 2.4 (Stationary induction [BH19])**.** *Let G be as in the notation and $\mu \in \mathrm{Prob}(G)$ any K-invariant admissible Borel probability measure. Let $\Gamma < G$ be any lattice and $\mu_0 \in \mathrm{Prob}(\Gamma)$ any Furstenberg measure.*

Let (M, ϕ) be any ergodic (Γ, μ_0)-von Neumann algebra. Set $\mathcal{M} = \mathrm{Ind}_\Gamma^G(M)$. Then there exists a μ-stationary normal state $\varphi \in \mathcal{M}_*$ so that $(\mathcal{M}, \varphi)$ is an ergodic (G, μ)-von Neumann algebra. Moreover, ϕ is Γ-invariant if and only if φ is G-invariant.

Proof. The following approach was suggested to us by Stefaan Vaes. His argument is shorter than our original proof (see [BH19, Section 4]). Thanks to Lemma 1.2, we may identify $\mathrm{L}^\infty(G/P, \nu_P)$ and $\mathrm{Har}^\infty(\Gamma, \mu_0)$ in a Γ-equivariant way. Define the normal ucp Γ-map $\theta : M \to \mathrm{Har}^\infty(\Gamma, \mu_0) : x \mapsto (\gamma \mapsto \phi(\sigma_\gamma^{-1}(x)))$. We may then regard θ as a normal ucp Γ-map $\theta : M \to \mathrm{L}^\infty(G/P, \nu_P)$. Inducing from Γ to G, we define the normal ucp G-map $\Theta : (\mathrm{L}^\infty(G) \mathbin{\overline{\otimes}} M)^\Gamma \to (\mathrm{L}^\infty(G) \mathbin{\overline{\otimes}} \mathrm{L}^\infty(G/P))^\Gamma$ by restricting $\mathrm{id}_{\mathrm{L}^\infty(G)} \otimes \theta$ to $(\mathrm{L}^\infty(G) \mathbin{\overline{\otimes}} M)^\Gamma$. Since $(G/P, \nu_P)$ is a G-space, it is easy to see that we can identify the G-von Neumann algebra $(\mathrm{L}^\infty(G) \mathbin{\overline{\otimes}} \mathrm{L}^\infty(G/P))^\Gamma$ with the G-von Neumann algebra $\mathrm{L}^\infty(G/\Gamma) \mathbin{\overline{\otimes}} \mathrm{L}^\infty(G/P)$ endowed with the diagonal action $G \curvearrowright G/\Gamma \times G/P$.

Since ν_P and $\mu * \nu_P$ are K-invariant Borel probability measures on G/P, it follows that $\nu_P = \mu * \nu_P$ and so ν_P is μ-stationary. Define $\varphi = (m_{G/\Gamma} \otimes \nu_P) \circ \Theta \in \mathcal{M}_*$. Since $m_{G/\Gamma}$ is G-invariant and ν_P is μ-stationary, φ is μ-stationary.

If ϕ is Γ-invariant, then $\theta(x) = \phi(x)\mathbf{1}$ for every $x \in M$. This implies that $\varphi = m_{G/\Gamma} \otimes \phi$ and so φ is G-invariant. Conversely, assume that φ is G-invariant. Let $x \in M$. For every $g \in G$, we have $g\nu_P(\theta(x)) = \nu_P(\theta(x))$. By [Fu62a, Theorem 5.3], since $(G/P, \nu_P)$ is the (G, μ)-Poisson boundary and since the bounded μ-harmonic function $G \to \mathbf{C} : g \mapsto g\nu_P(\theta(x))$ is constant, Lemma 1.2 implies that $\theta(x) \in \mathrm{L}^\infty(G/P, \nu_P)$ is an essentially constant function. By construction and since $(G/P, \nu_P)$ is the (Γ, μ_0)-Poisson boundary, Lemma 1.2 further implies that $\theta(x) = \phi(x)\mathbf{1}$. This finally shows that ϕ is Γ-invariant. □

Theorem 2.4 shows that in order to prove the structure theorem for stationary actions of higher rank lattices, we need to prove the structure theorem for stationary actions of the ambient Lie group. The structure theorem for stationary actions of higher rank simple Lie groups on arbitrary von Neumann algebras is a noncommutative analogue of Nevo–Zimmer's celebrated structure theorem [NZ00, Theorem 1]. This is the main topic of the next lecture.

3. Lecture 3: The noncommutative Nevo–Zimmer theorem

The main technical result of our work is the following noncommutative analogue of Nevo–Zimmer's celebrated structure theorem [NZ00, Theorem 1].

Theorem E (Noncommutative Nevo–Zimmer theorem [BH19])**.** *Let G be as in the notation. Let $\mu \in \mathrm{Prob}(G)$ be any K-invariant admissible Borel probability measure. Let $(\mathcal{M}, \varphi)$ be any ergodic (G, μ)-von Neumann algebra. Then the following dichotomy holds:*

- *Either φ is G-invariant.*
- *Or there exist a proper parabolic subgroup $P \subset Q \subsetneq G$ and a G-equivariant normal $\ast$-embedding $\Theta : \mathrm{L}^\infty(G/Q, \nu_Q) \to \mathcal{M}$ such that $\varphi \circ \Theta = \nu_Q$.*

To simplify the exposition, we present the proof of Theorem E in the particular case when $G = \mathrm{SL}_3(\mathbf{R})$. We refer the reader to [BH19, Section 5] for the proof in the general case.

Let us first introduce some notation and terminology. Define the following subgroups of $G = \mathrm{SL}_3(\mathbf{R})$:

$$P = \begin{pmatrix} \ast & \ast & \ast \\ 0 & \ast & \ast \\ 0 & 0 & \ast \end{pmatrix} \quad \overline{P} = \begin{pmatrix} \ast & 0 & 0 \\ \ast & \ast & 0 \\ \ast & \ast & \ast \end{pmatrix} \quad V = \begin{pmatrix} 1 & \ast & \ast \\ 0 & 1 & \ast \\ 0 & 0 & 1 \end{pmatrix} \quad \overline{V} = \begin{pmatrix} 1 & 0 & 0 \\ \ast & 1 & 0 \\ \ast & \ast & 1 \end{pmatrix}.$$

Let $(\mathcal{M}, \varphi)$ be any ergodic (G, μ)-von Neumann algebra with separable predual and denote by $\sigma : G \curvearrowright \mathcal{M}$ the corresponding continuous action. Note that $\varphi \in \mathcal{M}_\ast$ is a *faithful* normal state (see [BH19, Lemma 2.1]). If we denote by $(\pi_\varphi, H_\varphi, \xi_\varphi)$ the GNS triple associated with $(\mathcal{M}, \varphi)$, we may and will assume that $\pi_\varphi(x) = x$ for every $x \in \mathcal{M}$. Choose a globally G-invariant ultraweakly dense separable unital C*-subalgebra $\mathcal{A} \subset \mathcal{M}$ such that the action $G \curvearrowright \mathcal{A}$ is norm continuous (see the proof of [Ta03, Proposition XIII.1.2]). Since $(G/P, \nu_P)$ is the (G, μ)-Poisson boundary, we may define $\psi \in \mathcal{S}(\mathcal{A})$ as the unique P-invariant state corresponding to $\varphi|_{\mathcal{A}}$ so that $\varphi|_{\mathcal{A}} = \int_{G/P} \psi \circ \sigma_g^{-1} \, \mathrm{d}\nu_P(gP)$ (see Lemma 1.3 and [BH19, Theorem 2.6 and Example 2.7]).

Let $(\pi_\psi, H_\psi, \xi_\psi)$ be the GNS triple associated with $(\mathcal{A}, \psi)$ and set $\mathcal{N} = \pi_\psi(\mathcal{A})''$. We also denote by ψ the normal state $\langle \cdot \, \xi_\psi, \xi_\psi \rangle$ on $\mathcal{N}$. By definition, we have $\psi(\pi_\psi(a)) = \langle \pi_\psi(a)\xi_\psi, \xi_\psi \rangle = \psi(a)$ for every $a \in \mathcal{A}$. Since the action $P \curvearrowright \mathcal{A}$ is ψ-preserving, it extends to a continuous action $\sigma^{\mathcal{N}} : P \curvearrowright \mathcal{N}$ such that $\sigma_g^{\mathcal{N}}(\pi_\psi(a)) = \pi_\psi(\sigma_g(a))$, for all $g \in P$, $a \in \mathcal{A}$ (see [Ta02, Exercice I.10.7]). Denote by $q \in \mathcal{N}$ the support projection of $\psi \in \mathcal{N}_\ast$. Recall that q is the orthogonal projection of H_ψ onto the closure of $\mathcal{N}'\xi_\psi$. Since $\sigma^{\mathcal{N}} : P \curvearrowright \mathcal{N}$ is ψ-preserving, we have $q \in \mathcal{N}^P$. We point out that the action $\sigma^{\mathcal{N}} : P \curvearrowright \mathcal{N}$ need not be ergodic and so q need not be equal to 1. However, to simplify the exposition, we will assume that $q = 1$, that is, ψ is faithful on $\mathcal{N}$. We refer the reader to [BH19, Section 5] for the more general proof.

Our first task will be to embed G-equivariantly $\mathcal{M}$ into the induced von Neumann algebra of the action $P \curvearrowright \mathcal{N}$. Before doing so, let us give some concrete facts on this action, as in [NZ00, Section 7]. Following [Ma91, Lemma IV.2.2], the product map $\overline{V} \times P \to G : (\overline{v}, p) \mapsto \overline{v}p$ is a homeomorphism onto its image $\overline{V}P$ which is open and conull in G. As explained in [Ma91, IV.2.6], the restriction of the quotient map $G \to G/P : g \mapsto gP$ to $\overline{V}$ gives a measure space isomorphism $(\overline{V}, \nu_{\overline{V}}) \to (G/P, \nu_P) : \overline{v} \mapsto \overline{v}P$, whose inverse is denoted by τ. Observe that the Borel probability measure $\nu_{\overline{V}} \in \mathrm{Prob}(\overline{V})$ is in the same class as the Haar measure $m_{\overline{V}}$. Modifying τ on a set of measure 0, we can ensure that $\tau : G/P \to G$ is a measurable section to the quotient map $G \to G/P$ such that $\tau(\overline{v}P) = \overline{v}$, for all $\overline{v} \in \overline{V}$. We may identify the induced action $G \curvearrowright \mathrm{Ind}_P^G(\mathcal{N})$ with the continuous G-action $\widetilde{\sigma} : G \curvearrowright \mathrm{L}^\infty(G/P) \mathbin{\overline{\otimes}} \mathcal{N}$ given by the formula

$$\widetilde{\sigma}_g(F)(w) = \sigma^{\mathcal{N}}_{c_\tau(g, g^{-1}w)}(F(g^{-1}w)), \text{ for all } F \in \mathrm{L}^\infty(G/P) \mathbin{\overline{\otimes}} \mathcal{N}, g \in G, w \in G/P,$$

where $c_\tau : G \times G/P \to P$ is the measurable 1-cocycle associated with τ. By definition, we have $c_\tau(g, w) = \tau(gw)^{-1} g \tau(w)$ for all $g \in G$, $w \in G/P$. Furthermore, using our measure space identification, we have a von Neumann algebra isomorphism

$$\mathrm{L}^\infty(G/P) \mathbin{\overline{\otimes}} \mathcal{N} \cong \mathrm{L}^\infty(\overline{V}) \mathbin{\overline{\otimes}} \mathcal{N}.$$

The induced G-action may then be transported to a G-action on the right hand side von Neumann algebra. We can moreover obtain a very concrete formula when restricting the action to $\overline{P}$. Indeed, observe that

$$g\overline{v}P = \begin{cases} (g\overline{v})P \in \overline{V}P & \text{for all } g \in \overline{V}, \overline{v} \in \overline{V} \\ (g\overline{v}g^{-1})P \in \overline{V}P & \text{for all } g \in S, \overline{v} \in \overline{V}. \end{cases}$$

The above observation gives the corresponding 1-cocycle computation and we obtain that the $\overline{P}$-action $\widetilde{\sigma} : \overline{P} \curvearrowright \mathrm{L}^\infty(\overline{V}) \mathbin{\overline{\otimes}} \mathcal{N}$ is given by:

$$(\widetilde{\sigma}_g(F))(\overline{v}) = \begin{cases} F(g^{-1}\overline{v}) & \text{if } g \in \overline{V} \\ \sigma_g^{\mathcal{N}}(F(g^{-1}\overline{v}g)) & \text{if } g \in S \end{cases}, \text{ for all } F \in \mathrm{L}^\infty(\overline{V}) \mathbin{\overline{\otimes}} \mathcal{N}, \overline{v} \in \overline{V}.$$

Consider the map $\iota : \mathcal{A} \to \mathrm{L}^\infty(G/P) \mathbin{\overline{\otimes}} \mathcal{N}$ defined by the formula $\iota(a)(w) = \pi_\psi(\sigma_{\tau(w)}^{-1}(a))$, for all $a \in \mathcal{A}$, $w \in G/P$. Under the identification $\mathrm{L}^\infty(G/P) \mathbin{\overline{\otimes}} \mathcal{N} = \mathrm{L}^\infty(\overline{V}) \mathbin{\overline{\otimes}} \mathcal{N}$, the mapping $\iota : \mathcal{A} \to \mathrm{L}^\infty(\overline{V}) \mathbin{\overline{\otimes}} \mathcal{N}$ is given by the formula $\iota(a)(\overline{v}) = \pi_\psi(\sigma_{\overline{v}}^{-1}(a))$, for all $a \in \mathcal{A}$, $\overline{v} \in \overline{V}$.

Lemma 3.1. *The map ι extends to a well-defined G-equivariant normal unital $\ast$-embedding $\iota : \mathcal{M} \to \mathrm{L}^\infty(G/P) \mathbin{\overline{\otimes}} \mathcal{N}$ such that $(\nu_P \otimes \psi) \circ \iota = \varphi$.*

Proof. First, observe that for every $a \in \mathcal{A}$, $g \in G$ and $w \in G/P$, we have

$$\begin{aligned} \iota(\sigma_g(a))(w) &= \pi_\psi(\sigma_{\tau(w)}^{-1}(\sigma_g(a))) \\ &= \pi_\psi(\sigma_{\tau(w)^{-1}g\tau(g^{-1}w)\tau(g^{-1}w)^{-1}}(a)) \\ &= \sigma^{\mathcal{N}}_{c_\tau(g,g^{-1}w)}(\pi_\psi(\sigma_{\tau(g^{-1}w)}^{-1}(a))) = \widetilde{\sigma}_g(\iota(a))(w). \end{aligned}$$

Therefore $\iota(\sigma_g(a)) = \widetilde{\sigma}_g(\iota(a))$. Since $\psi \in \mathcal{N}_\ast$ is P-invariant, the bounded continuous map $G/P \to \mathbf{C} : gP \to \psi(\sigma_g^{-1}(a))$ is well-defined and we have

$$(\nu_P \otimes \psi)(\iota(a)) = \int_{G/P} \psi(\sigma_{\tau(w)}^{-1}(a))\, \mathrm{d}\nu_P(w) = \int_{G/P} \psi(\sigma_g^{-1}(a))\, \mathrm{d}\nu_P(gP) = \varphi(a), \text{ for all } a \in \mathcal{A}.$$

Thus, once we proved that $\iota : \mathcal{M} \to \mathrm{Ind}_P^G(\mathcal{N})$ extends to a normal unital $\ast$-embedding, we will necessarily have that ι is G-equivariant and $(\nu_P \otimes \psi) \circ \iota = \varphi$.

Set $H = \mathrm{L}^2(G/P, \nu_P) \otimes H_\psi$ and $\xi = 1_{G/P} \otimes \xi_\psi \in H$. Denote by $p \in \iota(\mathcal{A})' \cap \mathbf{B}(H)$ the orthogonal projection onto the closed linear span $K = \overline{\iota(\mathcal{A})\xi}$. We identify $\mathbf{B}(K) = p\mathbf{B}(H)p$. Observe that ξ is a $\iota(\mathcal{A})$-cyclic vector in K that implements the state φ on $\mathcal{A}$. Thus, by uniqueness of the GNS representation, the unitary representation $\mathcal{A} \to \mathbf{B}(K) : a \mapsto \iota(a)p$ is unitarily conjugate to $\pi_\varphi = \mathrm{id}$. In particular, it indeed extends to a normal unital $\ast$-isomorphism $\mathcal{M} \to \iota(\mathcal{A})''p : a \mapsto \iota(a)p$. We are left to check that the normal unital $\ast$-homomorphism $\iota(\mathcal{A})'' \to \iota(\mathcal{A})''p : f \mapsto fp$ is injective. Let $f \in \iota(\mathcal{A})''$ be such that $fp = 0$. For every $a \in \mathcal{A}$, we have $f\, \iota(a)\xi = 0$. Regarding $f \in \mathrm{L}^\infty(G/P, \mathcal{N})$, for every $a \in \mathcal{A}$ and almost every $w \in G/P$, we have $f(w)\, \pi_\psi(\sigma_{\tau(w)}^{-1}(a))\xi_\psi = 0$. Since $\mathcal{A}$ is separable, this implies that for almost every $w \in G/P$ and every $a \in \mathcal{A}$, we have $f(w)\, \pi_\psi(\sigma_{\tau(w)}^{-1}(a))\xi_\psi = 0$. Since ξ_ψ is $\pi_\psi(\mathcal{A})$-cyclic, we conclude that $f(w) = 0$ for almost every $w \in G/P$. This finally shows that $f = 0$. □

From now on, we will equally use the letter σ to denote any of the actions involved.

Proof of Theorem E. We assume that φ is not G-invariant and we will show that there exist a proper parabolic subgroup $P \subset Q \subsetneq G$ and a G-equivariant normal unital $\ast$-embedding $\Theta : \mathrm{L}^\infty(G/Q, \nu_Q) \to \mathcal{M}$ such that $\varphi \circ \Theta = \nu_Q$. In order to do this, we will construct an abelian G-invariant von Neumann subalgebra $\mathcal{Z}_0 \subset \mathcal{M}$ for which $\varphi|_{\mathcal{Z}_0}$ is not G-invariant. Then we will apply Nevo–Zimmer's result [NZ00, Theorem 1] to conclude.

Since φ is not G-invariant and since $\varphi \in \mathcal{M}_*$, $\varphi|_{\mathcal{A}}$ is not G-invariant either. Recall that $\varphi|_{\mathcal{A}} = \int_{G/P} \psi \circ \sigma_g^{-1} \, d\nu_P(gP)$. This implies that ψ is not G-invariant. Define the elementary subgroups

$$E_{21} = \begin{pmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad E_{31} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ * & 0 & 1 \end{pmatrix} \quad E_{32} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & * & 1 \end{pmatrix}.$$

Observe that $\overline{V} = \langle E_{21}, E_{31}, E_{32} \rangle$ and that $G = \langle P, \overline{V} \rangle$. Since ψ is P-invariant but not G-invariant, ψ is not $\overline{V}$-invariant. To fix the notation, we may and will assume that ψ is not E_{32}-invariant (the reasoning is analogous with E_{21} and E_{31}). Set

$$V_0 = \begin{pmatrix} 1 & * & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \overline{V}_0 = \begin{pmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ * & 0 & 1 \end{pmatrix} \quad \overline{U}_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & * & 1 \end{pmatrix} \quad s = \begin{pmatrix} \frac{1}{4} & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Then $\overline{U}_0 = E_{32}$, $\overline{V} = \overline{V}_0 \rtimes \overline{U}_0 = \overline{V}_0 \cdot \overline{U}_0$. Observe that s acts by conjugation on V_0 as a contracting automorphism. Likewise, s^{-1} acts by conjugation on $\overline{V}_0$ as a contracting automorphism. Note also that s commutes with $\overline{U}_0$. We identify $\mathrm{L}^\infty(\overline{V}) = \mathrm{L}^\infty(\overline{V}_0) \mathbin{\overline{\otimes}} \mathrm{L}^\infty(\overline{U}_0)$. The actions $s^{\mathbf{Z}} \curvearrowright \mathrm{L}^\infty(\overline{V}) \mathbin{\overline{\otimes}} \mathcal{N}$ and $\overline{V}_0 \curvearrowright \mathrm{L}^\infty(\overline{V}) \mathbin{\overline{\otimes}} \mathcal{N}$ are given by

$$\sigma_s(F)(\overline{v}_0, \overline{u}_0) = \sigma_s^{\mathcal{N}}(F(s^{-1}\overline{v}_0 s, \overline{u}_0)) \qquad \text{and} \qquad \sigma_g(F)(\overline{v}_0, \overline{u}_0) = F(g^{-1}\overline{v}_0, \overline{u}_0), \tag{3.1}$$

for all $F \in \mathrm{L}^\infty(\overline{V}_0 \rtimes \overline{U}_0) \mathbin{\overline{\otimes}} \mathcal{N}$, $g \in \overline{V}_0$, $\overline{v}_0 \in \overline{V}_0$, $\overline{u}_0 \in \overline{U}_0$.

The strategy of the **first part** of the proof of Theorem E is strongly inspired by the techniques developed by Nevo–Zimmer in [NZ00] (most notably the proofs of [NZ00, Theorem 1] and [NZ00, Proposition 10.1]). Since the action $\sigma : P \curvearrowright \mathcal{N}$ is ψ-preserving and since ψ is assumed to be faithful, we may consider the unique ψ-preserving faithful normal conditional expectation $\mathrm{E}_s : \mathcal{N} \to \mathcal{N}^s$.

Claim 3.2. Let $a \in \mathcal{A} \subset \mathcal{M}$. For every $n \in \mathbf{N}$, define $a_n = \frac{1}{n+1} \sum_{k=0}^n \sigma_s^k(a) \in \mathcal{A}$. The following assertions hold:

- For every $(\overline{v}_0, \overline{u}_0) \in \overline{V}$, $\iota(a_n)(\overline{v}_0, \overline{u}_0) \to \mathrm{E}_s(\iota(a)(\overline{e}, \overline{u}_0))$ strongly in $\mathcal{N}$.
- Define the bounded measurable function $f : \overline{V} \to \mathcal{N}^s : (\overline{v}_0, \overline{u}_0) \mapsto \mathrm{E}_s(\iota(a)(\overline{e}, \overline{u}_0))$. Then $f \in \iota(\mathcal{M})$. Thus, there exists a unique element $a_\infty \in \mathcal{M}$ such that $\iota(a_\infty) = f$ and $a_n \to a_\infty$ strongly in $\mathcal{M}$.

Proof of Claim 3.2. The proof follows the same strategy as the one of [NZ00, Proposition 7.1]. Let $(\overline{v}_0, \overline{u}_0) \in \overline{V} = \overline{V}_0 \rtimes \overline{U}_0$. Let $\varepsilon > 0$. Since $a \in \mathcal{A}$, since the action $G \curvearrowright \mathcal{A}$ is norm continuous and since $s^{-k}\overline{v}_0 s^k \to e$ in $\overline{V}$ as $k \to \infty$, there exists $k_0 = k_0(\overline{v}_0, \overline{u}_0) \in \mathbf{N}$ such that for every $k \geq k_0$, we have $\|\sigma_{s^{-k}\overline{v}_0^{-1}s^k}(a) - a\| \leq \varepsilon$. For all $k \geq k_0$, we have

$$\begin{aligned} \|\iota(\sigma_s^k(a))(\overline{v}_0, \overline{u}_0) - \iota(\sigma_s^k(a))(\overline{e}, \overline{u}_0)\| &= \|\pi_\psi(\sigma_{\overline{u}_0}^{-1}(\sigma_{\overline{v}_0}^{-1}(\sigma_s^k(a)))) - \pi_\psi(\sigma_{\overline{u}_0}^{-1}(\sigma_s^k(a)))\| \\ &\leq \|\sigma_{s^{-k}\overline{v}_0^{-1}s^k}(a) - a\| \\ &\leq \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we deduce that

$$\lim_n \|\iota(a_n)(\overline{v}_0, \overline{u}_0) - \iota(a_n)(\overline{e}, \overline{u}_0)\| = 0. \tag{3.2}$$

Since s commutes with $\overline{u}_0$, we have $\sigma_s^k(\iota(a)(\overline{e}, \overline{u}_0)) = \iota(\sigma_s^k(a))(\overline{e}, \overline{u}_0)$ for every $k \in \mathbf{N}$. Since the action $\sigma : s^{\mathbf{Z}} \curvearrowright \mathcal{N}$ is ψ-preserving, von Neumann's ergodic theorem implies that

$$\lim_{n \to \infty} \left\| \frac{1}{n+1} \sum_{k=0}^n \sigma_s^k(\iota(a)(\overline{e}, \overline{u}_0)) - \mathrm{E}_s(\iota(a)(\overline{e}, \overline{u}_0)) \right\|_\psi = 0.$$

This further implies that $\lim_n \|\iota(a_n)(\overline{e}, \overline{u}_0) - \mathrm{E}_s(\iota(a)(\overline{e}, \overline{u}_0))\|_\psi = 0$. Combining this with (3.2), we obtain that

$$\lim_n \|\iota(a_n)(\overline{v}_0, \overline{u}_0) - \mathrm{E}_s(\iota(a)(\overline{e}, \overline{u}_0))\|_\psi = 0. \tag{3.3}$$

This proves the first item. Define the bounded measurable function $f : \overline{V} \to \mathcal{N}^s : (\overline{v}_0, \overline{u}_0) \mapsto \mathrm{E}_s(\iota(a)(\overline{e}, \overline{u}_0))$. Observe that we may regard $f \in \mathrm{L}^\infty(\overline{U}_0, \mathcal{N}^s)$. Lebesgue's dominated convergence theorem implies that $\iota(a_n) \to f$ strongly in $\mathrm{L}^\infty(\overline{V}) \mathbin{\overline{\otimes}} \mathcal{N}$. This further implies that $f \in \iota(\mathcal{M})$. Set $a_\infty = \iota^{-1}(f) \in \mathcal{M}$. Then $a_n \to a_\infty$ strongly in $\mathcal{M}$. □

Define the following subgroups of $G = \mathrm{SL}_3(\mathbf{R})$:

$$W_{0,s} = \left\{ \begin{pmatrix} \frac{1}{4}^n & \alpha & \beta \\ 0 & 2^n & 0 \\ 0 & 0 & 2^n \end{pmatrix} \mid n \in \mathbf{Z}, \alpha, \beta \in \mathbf{R} \right\} \quad \text{and} \quad P_0 = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix}.$$

It follows from the above decomposition of P_0 that $W_{0,s}$ is a normal subgroup of P_0. Since $W_{0,s} \subset P \subset P_\theta$, it follows that $W_{0,s}$ is a normal subgroup of P. We have the following inclusions of fixed-point von Neumann subalgebras:

$$\mathcal{N}^P \subset \mathcal{N}^{W_{0,s}} \subset \mathcal{N}^s \subset \mathcal{N}.$$

Observe that $\mathcal{N}^{W_{0,s}}$ is a P-von Neumann algebra since $W_{0,s}$ is a normal subgroup of P. Mautner's phenomenon implies that $\mathcal{N}^{W_{0,s}} = \mathcal{N}^s$. Indeed, since the action $\sigma : P \curvearrowright \mathcal{N}$ is ψ-preserving, for every $b \in \mathcal{N}^s$ and every $v_0 \in V_0$, we have

$$\|\sigma_{v_0}(b) - b\|_\psi = \|\sigma_{s^k v_0 s^{-k}}(b) - b\|_\psi \to 0 \quad \text{as} \quad k \to \infty.$$

This implies that $\sigma_{v_0}(b) = b$. Therefore, $\mathcal{N}^s = \mathcal{N}^{W_{0,s}}$ is a P-von Neumann algebra. Denote by $m_{\overline{U}_0}$ (resp. $m_{\overline{V}_0}$) a Haar measure on $\overline{U}_0$ (resp. $\overline{V}_0$).

Define $\mathcal{M}_0 = \iota^{-1}(\mathrm{L}^\infty(G/P) \mathbin{\overline{\otimes}} \mathcal{N}^s)$. Since $\mathcal{N}^s \subset \mathcal{N}$ is globally P-invariant and since ι is a G-equivariant normal unital $*$-embedding, it follows that $\mathcal{M}_0 \subset \mathcal{M}$ is a globally G-invariant von Neumann subalgebra. Since the action $G \curvearrowright \mathcal{M}$ is ergodic, the action $G \curvearrowright \mathcal{M}_0$ is ergodic.

Claim 3.3. The action $G \curvearrowright \mathcal{M}_0$ is not φ-preserving. In particular, we have $\mathcal{M}_0 \neq \mathbf{C}1$.

Proof of Claim 3.3. By contradiction, assume that the action $G \curvearrowright \mathcal{M}_0$ is φ-preserving. Since ι is a G-equivariant normal unital $*$-embedding such that $(\nu_P \otimes \psi) \circ \iota = \varphi$ and since the continuous P-action $\sigma : P \curvearrowright \mathcal{N}^s$ is ψ-preserving, for every $x \in \mathcal{M}_0$, the following quantity does not depend on $g \in G$:

$$\begin{aligned} \varphi(\sigma_g^{-1}(x)) &= \int_{G/P} \psi(\iota(\sigma_g^{-1}(x))(w)) \, \mathrm{d}\nu_P(w) \\ &= \int_{G/P} \psi(\sigma_{c_\tau(g^{-1}, gw)}(\iota(x)(gw))) \, \mathrm{d}\nu_P(w) \\ &= \int_{G/P} \psi(\iota(x)(gw)) \, \mathrm{d}\nu_P(w). \end{aligned}$$

Therefore, the bounded μ-harmonic function $G \to \mathbf{C} : g \mapsto \int_{G/P} \psi(\iota(x)(gw)) \, \mathrm{d}\nu_P(w)$ is constant. Since $(G/P, \nu_P)$ is the (G, μ)-Poisson boundary, Theorem 1.2 implies that the bounded measurable function $G/P \to \mathbf{C} : w \mapsto \psi(\iota(x)(w))$ is ν_P-almost everywhere constant.

Let $a \in \mathcal{A}$ be any element. Regarding $\mathrm{L}^\infty(G/P, \nu_P) = \mathrm{L}^\infty(\overline{V}, \nu_{\overline{V}})$, since $a_\infty \in \mathcal{M}_0$ and since $\iota(a_\infty) \in \mathrm{L}^\infty(\overline{U}_0) \mathbin{\overline{\otimes}} \mathcal{N}^s$ by Claim 3.2, this implies that the bounded measurable function $\overline{U}_0 \to \mathbf{C} : \overline{u}_0 \mapsto \psi(\iota(a_\infty)(\overline{u}_0))$ is $m_{\overline{U}_0}$-almost everywhere constant. By Claim 3.2, for $m_{\overline{U}_0}$-almost every $\overline{u}_0 \in \overline{U}_0$, $\iota(a_n)(\overline{e}, \overline{u}_0) \to \iota(a_\infty)(\overline{u}_0)$ strongly in $\mathcal{N}$. Since the action $\sigma : s^{\mathbf{Z}} \curvearrowright \mathcal{N}$ is ψ-preserving and since s commutes with $\overline{U}_0$, we have $\psi(\iota(a_n)(\overline{e}, \overline{u}_0)) = \psi(\iota(a)(\overline{e}, \overline{u}_0)) = \psi(\sigma_{\overline{u}_0}^{-1}(a))$ for every $n \in \mathbf{N}$ and every $\overline{u}_0 \in \overline{U}_0$. Then $\psi(\iota(a_\infty)(\overline{u}_0)) = \lim_n \psi(\iota(a_n)(\overline{e}, \overline{u}_0)) = \psi(\sigma_{\overline{u}_0}^{-1}(a))$ for $m_{\overline{U}_0}$-almost every $\overline{u}_0 \in \overline{U}_0$. Since the function $\overline{U}_0 \to \mathbf{C} : \overline{u}_0 \mapsto \psi(\sigma_{\overline{u}_0}^{-1}(a))$ is continuous and constant $m_{\overline{U}_0}$-almost everywhere, it is constant everywhere. This shows that for every $\overline{u}_0 \in \overline{U}_0$ and every $a \in \mathcal{A}$, $\psi(\sigma_{\overline{u}_0}^{-1}(a)) = \psi(a)$. This contradicts the fact that ψ is not $\overline{U}_0$-preserving. □

The strategy of the **second part** of the proof of Theorem E is new compared to the proof of [NZ00, Theorem 1]. We make use of various von Neumann algebraic techniques involving essential values into noncommutative algebras, disintegration theory and the slice mapping theorem of Ge–Kadison [GK95] and its generalization by Strătilă–Zsidó [SZ98]. We will show that the continuous G-action $G \curvearrowright \mathcal{Z}(\mathcal{M}_0)$ (which is ergodic and μ-stationary) does not preserve φ. We will then apply [NZ00, Theorem 1] to obtain the second item stated in Theorem E.

Recall that $\overline{V} = \overline{V}_0 \rtimes \overline{U}_0$ so that $\mathrm{L}^\infty(\overline{V}) = \mathrm{L}^\infty(\overline{V}_0) \mathbin{\overline{\otimes}} \mathrm{L}^\infty(\overline{U}_0)$. Choose a Borel probability measure $\nu_{\overline{V}_0} \in \mathrm{Prob}(\overline{V}_0)$ that is in the same class as the Haar measure $m_{\overline{V}_0}$. Put $\mathcal{Q} = \mathrm{L}^\infty(\overline{U}_0) \mathbin{\overline{\otimes}} \mathcal{N}^s$ and regard

$$\mathrm{L}^\infty(\overline{V}) \mathbin{\overline{\otimes}} \mathcal{N}^s = \mathrm{L}^\infty(\overline{V}_0) \mathbin{\overline{\otimes}} \left(\mathrm{L}^\infty(\overline{U}_0) \mathbin{\overline{\otimes}} \mathcal{N}^s\right) = \mathrm{L}^\infty(\overline{V}_0) \mathbin{\overline{\otimes}} \mathcal{Q} = \mathrm{L}^\infty(\overline{V}_0, \mathcal{Q}).$$

As usual, we endow $\mathcal{Q}$ with the strong operator topology. Let $f \in \mathrm{L}^\infty(\overline{V}_0, \mathcal{Q})$. Up to discarding a null measurable subset in $\overline{V}_0$, we may view $f \in \mathrm{L}^\infty(\overline{V}_0, \mathrm{Ball}_{\mathcal{Q}}(0, \|f\|_\infty))$. Denote by $F_f \subset \mathrm{Ball}_{\mathcal{Q}}(0, \|f\|_\infty)$ the essential range of f. Since $\mathcal{Q}$ has separable predual, $\mathrm{Ball}_{\mathcal{Q}}(0, \|y\|_\infty) \subset \mathcal{Q}$ is Polish with respect to the strong operator topology. Then the essential range $F_f \subset \mathrm{Ball}_{\mathcal{Q}}(0, \|f\|_\infty)$ of f is strongly closed in $\mathcal{Q}$ and coincides with the set of essential values of f. Moreover, up to discarding a null measurable subset of $\overline{V}_0$, we may view $f \in \mathrm{L}^\infty(\overline{V}_0, F_f)$.

Regard $\iota(\mathcal{M}_0) \subset \mathrm{L}^\infty(\overline{V}_0, \mathcal{Q})$. Choose a strongly dense countable subset $\{x_n \mid n \in \mathbf{N}\}$ of $\mathcal{M}_0$. Define the von Neumann subalgebra $\mathcal{Q}_0 \subset \mathcal{Q}$ by

$$\mathcal{Q}_0 = \bigvee \{F_{\iota(x_n)} \mid n \in \mathbf{N}\}.$$

Claim 3.4. We have $\mathbf{C}1_{\overline{V}_0} \mathbin{\overline{\otimes}} \mathcal{Q}_0 \subset \iota(\mathcal{M}_0) \subset \mathrm{L}^\infty(\overline{V}_0) \mathbin{\overline{\otimes}} \mathcal{Q}_0$.

Proof of Claim 3.4. Up to discarding a null measurable subset of $\overline{V}_0$, we may assume that $\iota(x_n) \in \mathrm{L}^\infty(\overline{V}_0, F_{\iota(x_n)})$ for every $n \in \mathbf{N}$. This implies that $\iota(x_n) \in \mathrm{L}^\infty(\overline{V}_0, \mathcal{Q}_0)$ for every $n \in \mathbf{N}$. Since $\{x_n \mid n \in \mathbf{N}\}$ is strongly dense in $\mathcal{M}_0$ and since $\mathcal{Q}_0$ is a von Neumann algebra, this further implies that $\iota(\mathcal{M}_0) \subset \mathrm{L}^\infty(\overline{V}_0, \mathcal{Q}_0) = \mathrm{L}^\infty(\overline{V}_0) \mathbin{\overline{\otimes}} \mathcal{Q}_0$.

Choose a faithful state $\Psi \in (\mathcal{Q}_0)_*$. Let $n \in \mathbf{N}$ and set $f = \iota(x_n) \in \iota(\mathcal{M}_0)$. Regard $f \in \mathrm{L}^\infty(\overline{V}_0, F_f)$ with $F_f \subset \mathrm{Ball}_{\mathcal{Q}_0}(0, \|f\|_\infty)$. Let $b \in F_f$ be any essential value of f. For every $k \in \mathbf{N}$, define

$$\overline{B}_k = \left\{\overline{v}_0 \in \overline{V}_0 \mid \|f(\overline{v}_0) - b\|_\Psi < \frac{1}{k+1}\right\}.$$

Since $b \in F_f$, we have $\nu_{\overline{V}_0}(\overline{B}_k) > 0$. For every $k \in \mathbf{N}$, applying [Ma91, Lemma IV.2.5(a)] to $\overline{B}_k$, there exist $\overline{h}_k \in \overline{V}_0$ and $n_k \in \mathbf{N}$ large enough so that $\lim_k \nu_{\overline{V}_0}(s^{n_k}\, \overline{h}_k \overline{B}_k\, s^{-n_k}) = 1$. For every $k \in \mathbf{N}$, set $g_k = s^{n_k} \overline{h}_k \in s^{\mathbf{Z}} \ltimes \overline{V}_0$. We will show that $\sigma_{g_k}(f)$ converges strongly to $1_{\overline{V}_0} \otimes b$ in $\mathcal{Q}_0$ as $k \to \infty$. This will clearly imply that $1_{\overline{V}_0} \otimes b \in \iota(\mathcal{M}_0)$ and the claim will follow.

In view of formula (3.1) and since s acts trivially on $\mathcal{N}^s$, observe that

$$\sigma_{g_k}(f)(\overline{v}_0) = \sigma_s^{n_k}(f((\overline{h}_k)^{-1} s^{-n_k} \overline{v}_0 s^{n_k})) = f((\overline{h}_k)^{-1} s^{-n_k} \overline{v}_0 s^{n_k}), \text{ for all } \overline{v}_0 \in \overline{V}_0.$$

For all $k \in \mathbf{N}$, set $\overline{C}_k = s^{n_k}\, \overline{h}_k \overline{B}_k\, s^{-n_k} \subset \overline{V}_0$. By assumption, we have $\lim_k \nu_{\overline{V}_0}(\overline{C}_k) = 1$ and we may compute

$$\begin{aligned}
\|\sigma_{g_k}(f) - 1_{\overline{V}_0} \otimes b\|^2_{\nu_{\overline{V}_0} \otimes \Psi} &= \int_{\overline{V}_0} \|f((\overline{h}_k)^{-1} s^{-n_k} \overline{v}_0 s^{n_k}) - b\|_\Psi^2 \, \mathrm{d}\nu_{\overline{V}_0}(\overline{v}_0) \\
&= \int_{\overline{C}_k} \|f((\overline{h}_k)^{-1} s^{-n_k} \overline{v}_0 s^{n_k}) - b\|_\Psi^2 \, \mathrm{d}\nu_{\overline{V}_0}(\overline{v}_0) \\
&\quad + \int_{\overline{V}_0 \setminus \overline{C}_k} \|f((\overline{h}_k)^{-1} s^{-n_k} \overline{v}_0 s^{n_k}) - b\|_\Psi^2 \, \mathrm{d}\nu_{\overline{V}_0}(\overline{v}_0) \\
&\leq \frac{1}{(k+1)^2}\, \nu_{\overline{V}_0}(\overline{C}_k) + 2\|f\|_\infty\, \nu_{\overline{V}_0}(\overline{V}_0 \setminus \overline{C}_k).
\end{aligned}$$

This quantity converges to 0 as $k \to \infty$. Since the sequence $(\sigma_{g_k}(f))_{k \in \mathbf{N}}$ is uniformly bounded, it follows that $\sigma_{g_k}(f) \to 1_{\overline{V}_0} \otimes b$ strongly as $k \to \infty$. ☐

Claim 3.4 implies that $\mathbf{C}1_{\overline{V}_0} \mathbin{\overline{\otimes}} \mathcal{Z}(\mathcal{Q}_0) \subset \mathcal{Z}(\iota(\mathcal{M}_0)) \subset \mathrm{L}^\infty(\overline{V}_0) \mathbin{\overline{\otimes}} \mathcal{Z}(\mathcal{Q}_0)$.

Claim 3.5. We have that $\mathbf{C}1_{\overline{V}_0} \mathbin{\overline{\otimes}} \mathcal{Z}(\mathcal{Q}_0) \neq \mathcal{Z}(\iota(\mathcal{M}_0))$.

Proof of Claim 3.5. The following approach was suggested to us by Narutaka Ozawa. While our original proof relies on Ge–Kadison's splitting theorem (see [GK95, Theorem A]), his argument relies on Strătilă–Zsidó's generalization of Ge–Kadison's result (see [SZ98, Theorem 4.1]). By contradiction, assume that $\mathbf{C}1_{\overline{V}_0} \mathbin{\overline{\otimes}} \mathcal{Z}(\mathcal{Q}_0) = \mathcal{Z}(\iota(\mathcal{M}_0))$. Since

$$\iota(\mathcal{M}_0) \cap (\mathrm{L}^\infty(\overline{V}_0) \mathbin{\overline{\otimes}} \mathcal{Z}(\mathcal{Q}_0)) = \mathcal{Z}(\iota(\mathcal{M}_0)) = \mathbf{C}1_{\overline{V}_0} \mathbin{\overline{\otimes}} \mathcal{Z}(\mathcal{Q}_0)$$

splits, [SZ98, Theorem 4.1] implies that $\iota(\mathcal{M}_0) = \mathbf{C}1_{\overline{V}_0} \mathbin{\overline{\otimes}} \mathcal{Q}_0$ splits. Since s acts trivially on $\mathcal{Q}_0$, (3.1) implies that s acts trivially on $\mathcal{M}_0$. Since G is a connected simple Lie group with trivial center, this implies that G acts trivially on $\mathcal{M}_0$. This contradicts Claim 3.3. ☐

Claim 3.6. The action $G \curvearrowright \mathcal{Z}(\mathcal{M}_0)$ is not φ-preserving.

Proof of Claim 3.6. By contradiction, assume that the action $G \curvearrowright \mathcal{Z}(\mathcal{M}_0)$ is φ-preserving. Regard $\iota(\mathcal{Z}(\mathcal{M}_0)) \subset \iota(\mathcal{M}_0) \subset \mathrm{L}^\infty(\overline{V}) \mathbin{\overline{\otimes}} \mathcal{N}^s$. The same argument as in the proof of Claim 3.3 shows that for every $x \in \mathcal{Z}(\mathcal{M}_0)$, the bounded measurable function $\overline{V} \to \mathbf{C} : \overline{v} \mapsto \psi(\iota(x)(\overline{v}))$ is $\nu_{\overline{V}}$-almost everywhere constant. Since $(\nu_{\overline{V}} \otimes \psi) \circ \iota = \varphi$, this means that for every $x \in \mathcal{Z}(\mathcal{M}_0)$, we have $(\mathrm{id}_{\overline{V}} \otimes \psi)(\iota(x)) = \varphi(x)\, 1_{\overline{V}}$.

Recall that $\mathrm{L}^\infty(\overline{V}) = \mathrm{L}^\infty(\overline{V}_0) \mathbin{\overline{\otimes}} \mathrm{L}^\infty(\overline{U}_0)$, $1_{\overline{V}} = 1_{\overline{V}_0} \otimes 1_{\overline{U}_0}$, $\mathrm{id}_{\overline{V}} = \mathrm{id}_{\overline{V}_0} \otimes \mathrm{id}_{\overline{U}_0}$. Once again, consider the splitting $\mathrm{L}^\infty(\overline{V}) \mathbin{\overline{\otimes}} \mathcal{N}^s = \mathrm{L}^\infty(\overline{V}_0) \mathbin{\overline{\otimes}} (\mathrm{L}^\infty(\overline{U}_0) \mathbin{\overline{\otimes}} \mathcal{N}^s)$. Choose a Borel probability measure $\nu_{\overline{U}_0} \in \mathrm{Prob}(\overline{U}_0)$ that is in the same class as the Haar measure $m_{\overline{U}_0}$. Set $\psi_0 = \nu_{\overline{U}_0} \otimes \psi \in (\mathrm{L}^\infty(\overline{U}_0) \mathbin{\overline{\otimes}} \mathcal{N}^s)_*$ and observe that ψ_0 is a faithful normal state on $\mathrm{L}^\infty(\overline{U}_0) \mathbin{\overline{\otimes}} \mathcal{N}^s$. Then we obtain

$$\begin{aligned}
\forall x \in \mathcal{Z}(\mathcal{M}_0), \quad (\mathrm{id}_{\overline{V}_0} \otimes \psi_0)(\iota(x)) &= (\mathrm{id}_{\overline{V}_0} \otimes \nu_{\overline{U}_0} \otimes \psi)(\iota(x)) \\
&= (\mathrm{id}_{\overline{v}_0} \otimes \nu_{\overline{U}_0}) \left((\mathrm{id}_{\overline{V}_0} \otimes \mathrm{id}_{\overline{U}_0} \otimes \psi)(\iota(x)) \right) \\
&= (\mathrm{id}_{\overline{V}_0} \otimes \nu_{\overline{U}_0}) \left(\varphi(x)\, 1_{\overline{V}} \right) \\
&= \varphi(x)\, 1_{\overline{V}_0}.
\end{aligned}$$

Observe that $\iota(\mathcal{Z}(\mathcal{M}_0)) = \mathcal{Z}(\iota(\mathcal{M}_0))$. Since $\mathbf{C}1_{\overline{V}_0} \mathbin{\overline{\otimes}} \mathcal{Z}(\mathcal{Q}_0) \subset \iota(\mathcal{Z}(\mathcal{M}_0))$, for every $x \in \mathcal{Z}(\mathcal{M}_0)$ and every $b \in \mathcal{Z}(\mathcal{Q}_0)$, we have $\iota(x)(1_{\overline{V}_0} \otimes b) \in \iota(\mathcal{Z}(\mathcal{M}_0))$ and so $(\mathrm{id}_{\overline{V}_0} \otimes b\psi_0)(\iota(x)) = (\mathrm{id}_{\overline{V}_0} \otimes \psi_0)(\iota(x)(1_{\overline{V}_0} \otimes b)) \in \mathbf{C}1_{\overline{V}_0}$. Since $\psi_0|_{\mathcal{Z}(\mathcal{Q}_0)}$ is faithful, Hahn–Banach theorem implies that the linear subspace $\{b\psi_0 \mid b \in \mathcal{Z}(\mathcal{Q}_0)\}$ is $\|\cdot\|$-dense in $\mathcal{Z}(\mathcal{Q}_0)_*$. This implies that for every $x \in \mathcal{Z}(\mathcal{M}_0)$ and every $\rho \in \mathcal{Z}(\mathcal{Q}_0)_*$, we have $(\mathrm{id}_{\overline{V}_0} \otimes \rho)(\iota(x)) \in \mathbf{C}1_{\overline{V}_0}$. Moreover, for every $x \in \mathcal{Z}(\mathcal{M}_0)$ and every $\rho \in \mathrm{L}^\infty(\overline{V}_0)_*$, we have $(\rho \otimes \mathrm{id}_{\mathcal{Z}(\mathcal{Q}_0)})(\iota(x)) \in \mathcal{Z}(\mathcal{Q}_0)$. Then [GK95, Theorem B] implies that $\iota(\mathcal{Z}(\mathcal{M}_0)) = \mathbf{C}1_{\overline{V}_0} \mathbin{\overline{\otimes}} \mathcal{Z}(\mathcal{Q}_0)$ and thus $\mathcal{Z}(\iota(\mathcal{M}_0)) = \mathbf{C}1_{\overline{V}_0} \mathbin{\overline{\otimes}} \mathcal{Z}(\mathcal{Q}_0)$. This contradicts Claim 3.5. ☐

By Claim 3.6, we have that $(\mathcal{Z}(\mathcal{M}_0), \varphi)$ is an abelian ergodic (G, μ)-von Neumann algebra for which the action $G \curvearrowright \mathcal{Z}(\mathcal{M}_0)$ is not φ-preserving. We can now apply [NZ00, Theorem 1] to obtain that there exist a proper parabolic subgroup $P \subset Q \subsetneq G$ and a G-equivariant normal unital $*$-embedding $\Theta : \mathrm{L}^\infty(G/Q) \to \mathcal{Z}(\mathcal{M}_0) \subset \mathcal{M}$ such that $\varphi \circ \Theta = \nu_Q$. This finishes the proof of Theorem E. ☐

The proof of Theorem D is now a combination of Theorem E, Theorem 2.4 and a disintegration argument for $*$-representations of separable unital C*-algebras. We refer the reader to [BH19, Section 6] for the details.

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