Ergodic group theory

M2 AAG, Université Paris-Saclay, Orsay, 2020-2021

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Abstract. These are the lecture notes of a graduate course on *ergodic group theory* given at Université Paris-Saclay, Orsay, during 2020-2021. In this course, we introduce various tools from group theory, ergodic theory and functional analysis to study the structure of discrete groups that arise as lattices in locally compact groups. Topics include: locally compact groups and their lattices; group actions on measure spaces; unitary representations; induction; amenability; Howe–Moore property; Kazhdan's property (T); stationary measures; Poisson boundaries; reduced cohomology. The aim of the course is to prove Bader–Shalom's normal subgroup theorem for irreductible lattices in product groups that can be regarded as an extension of Margulis's celebrated normal subgroup theorem for lattices in semisimple Lie groups.

Contents

CHAPTER 1

Locally compact groups and lattices

In this chapter, we introduce basic properties of locally compact groups and their lattices. We show that $\mathrm{SL}_d(\mathbb{Z})$ is a lattice in $\mathrm{SL}_d(\mathbb{R})$ for every $d \geq 2$.

1. Generalities on locally compact groups

DEFINITION 1.1. Let G be any group endowed with a Hausdorff topology. We say that G is a *topological group* if the map $G \times G \to G : (g, h) \mapsto gh^{-1}$ is continuous. We then say that G is *locally compact* if there exists a compact neighborhood $U \subset G$ of the identity element $e \in G$.

Let G be any locally compact group. We say that G is

- *first countable* if there exists a countable neighborhood basis of $e \in G$.
- *second countable* if there exists a countable basis for the topology on G.
- *σ-compact* if there exists an increasing sequence of compact subsets $Q_n \subset G$ such that $G = \bigcup_{n \in \mathbb{N}} Q_n$.
- *compactly generated* if there exists a compact subset $Q \subset G$ such that $e \in Q$ and $G = \bigcup_{n \geq 1} Q^n$.
- *totally disconnected* if the connected component of $e \in G$ is equal to $\{e\}$.

The identity element $e \in G$ has a neighborhood basis consisting of compact subsets (see [**DE14**, Corollary A.8.2]). Any open subgroup $H < G$ is also closed since $G \setminus H = \bigcup_{gH \neq H} gH$. Any compactly generated group G is σ-compact. Any locally compact group G has a compactly generated open subgroup $H < G$. Indeed, choose a compact neighborhood $U \subset G$ of $e \in G$. Then $H \doteq \bigcup_{n \geq 1} (U \cup U^{-1})^n$ is a compactly generated open subgroup of G. In particular, any *connected* locally compact group is compactly generated. A locally compact group G is second countable if and and only it is first countable and σ-compact (see [**St73**]). Any locally compact second countable group G is metrizable with a proper left invariant metric (see [**St73, HP06**]).

The class of locally compact groups is stable under taking closed subgroups, finite direct products and quotients with respect to closed normal subgroups. More precisely, we record the following facts.

PROPOSITION 1.2. *The following assertions hold:*

(i) *If G is a locally compact group and $H \leq G$ is a closed subgroup, then H endowed with the induced topology is locally compact.*
(ii) *If $d \geq 1$ and $G_1, \dots, G_d$ are locally compact groups, then the product group $G = G_1 \times \cdots \times G_d$ endowed with the product topology is locally compact.*
(iii) *If G is a locally compact group and $N \lhd G$ is a closed normal subgroup, the quotient group G/N endowed with the quotient topology is locally compact.*
(iv) *If G is a locally compact group acting continuously on a locally compact group H by continuous automorphisms, then the semi-direct product group $G \ltimes H$ endowed with the product topology is locally compact.*

The proof of Proposition 1.2 is left to the reader as an exercise.

Examples 1.3. Here are some examples of locally compact groups. Let $d \geq 1$.

(i) Any group G endowed with the discrete topology is locally compact. In these notes, any countable group will always be endowed with its discrete topology.
(ii) Any compact group K is locally compact. In particular, the following compact groups

$$\mathbb{T}^d \doteq \left\{(z_1, \dots, z_d) \in \mathbb{C}^d \mid \forall 1 \leq i \leq d, |z_i| = 1\right\}$$
$$\mathrm{SO}_d(\mathbb{R}) \doteq \{A \in \mathrm{SL}_d(\mathbb{R}) \mid A^*A = AA^* = 1_d\}$$
$$\mathcal{U}(d) \doteq \{A \in \mathrm{GL}_d(\mathbb{C}) \mid A^*A = AA^* = 1_d\}$$

are locally compact.
(iii) Any (finite dimensional) real Lie group G is locally compact.
  - The abelian group $(\mathbb{R}^d, +)$ endowed with the usual topology is locally compact.
  - The *general linear group* $\mathrm{GL}_d(\mathbb{R})$ can be regarded as the open (dense) subset of invertible matrices in $\mathbb{M}_d(\mathbb{R}) \cong \mathbb{R}^{d^2}$. Endowed with the topology coming from $\mathbb{R}^{d^2}$, the group $\mathrm{GL}_d(\mathbb{R})$ is locally compact.
  - The *special linear group* $\mathrm{SL}_d(\mathbb{R}) = \ker(\det)$ is a closed subgroup of $\mathrm{GL}_d(\mathbb{R})$ and so $\mathrm{SL}_d(\mathbb{R})$ is locally compact.
  - The semi-direct product group $\mathrm{SL}_d(\mathbb{R}) \ltimes \mathbb{R}^d$ is locally compact.
(iv) Any (finite dimensional) p-adic Lie group G is totally disconnected locally compact. In particular, for every prime $p \in \mathcal{P}$, the groups $\mathrm{GL}_d(\mathbb{Q}_p)$ and $\mathrm{SL}_d(\mathbb{Q}_p)$ are totally disconnected locally compact.
(v) Let $\mathsf{T} = (\mathsf{V}, \mathsf{E})$ be any locally finite tree and denote by $\mathrm{Aut}(\mathsf{T})$ the automorphism group of T. Endowed with the topology of pointwise convergence, the group $\mathrm{Aut}(\mathsf{T})$ is totally disconnected locally compact.

Let X be any locally compact space, meaning that every $x \in X$ has a compact neighborhood. We denote by $\mathcal{B}(X)$ the σ-algebra of Borel subsets of X. We say that a Borel measure ν on X, that is, a measure defined on $\mathcal{B}(X)$ is *regular* if the following conditions are satisfied:

(i) For every Borel subset $B \subset X$, we have

$$\nu(B) = \inf \{\nu(V) \mid V \text{ is open and } B \subset V\}.$$

(ii) For every open subset $U \subset X$, we have

$$\nu(U) = \sup \{\nu(K) \mid K \text{ is compact and } K \subset U\}.$$

(iii) For every compact subset $K \subset X$, we have $\nu(K) < +\infty$.

When ν is nonzero, define the *support* of ν by

$$\operatorname{supp}(\nu) = \bigcap \{F \mid F \subset X \text{ is closed and } \nu(X \setminus F) = 0\}.$$

Observe that $\operatorname{supp}(\nu)$ is closed and $\nu(X \setminus \operatorname{supp}(\nu)) = 0$.

If any open subset of X is σ-compact, then any Borel measure on X that satisfies condition (iii) is regular (see [**Ru87**, Theorem 2.18]). In particular, using [**DE14**, Lemma A.8.1(i)], if X is a locally compact second countable space, then any open subset of X is σ-compact and thus any Borel measure on X that satisfies condition (iii) is regular.

Denote by $\mathrm{C}_c(X)$ the space of compactly supported continuous functions on X. We say that a linear functional $\Phi : \mathrm{C}_c(X) \to \mathbb{C}$ is *positive* if $\Phi(f) \geq 0$ for every $f \in \mathrm{C}_c(X)_+$. By Riesz's representation theorem (see [**Ru87**, Theorem 2.14]), for every positive linear functional $\Phi : \mathrm{C}_c(X) \to \mathbb{C}$, there exists a unique regular Borel measure ν on X such that

$$\forall f \in \mathrm{C}_c(X), \quad \Phi(f) = \int_X f(x) \, \mathrm{d}\nu(x).$$

In that case, we will simply write $\Phi = \nu$. Note that for every regular Borel measure ν on X and every $p \in [1, +\infty)$, the space $\mathrm{C}_c(X)$ is $\|\cdot\|_p$-dense in the Banach space $\mathrm{L}^p(X, \nu)$ of all ν-equivalence classes of p-integrable functions on X.

THEOREM 1.4 (Haar). *Let G be any locally compact group. Then there exists a nonzero regular Borel measure m_G on G that is unique up to multiplicative constant and that satisfies one of the following equivalent conditions:*

(i) *For every Borel subset $B \subset G$ and every $g \in G$, $m_G(gB) = m_G(B)$.*

(ii) *For every $f \in \mathrm{C}_c(G)$ and every $g \in G$,*

$$\int_G f(g^{-1}h) \, \mathrm{d}m_G(h) = \int_G f(h) \, \mathrm{d}m_G(h)$$

We say that m_G is a left invariant Haar measure *on G.*

For a proof of Theorem 1.4, we refer the reader to [**HR79**, Chapter 15]. The locally compact group G is σ-compact if and only if the left invariant Haar measure m_G is σ-finite.

Theorem 1.4 also implies that exists a nonzero regular Borel measure μ_G on G that is unique up to multiplicative constant and that satisfies one of the following equivalent conditions:

(i) For every Borel subset $B \subset G$ and every $g \in G$, $\mu_G(Bg) = \mu_G(B)$.
(ii) For every $f \in \mathrm{C}_c(G)$ and every $g \in G$,

$$\int_G f(hg)\, \mathrm{d}\mu_G(h) = \int_G f(h)\, \mathrm{d}\mu_G(h)$$

We say that μ_G is a *right invariant Haar measure* on G. Indeed, any left invariant Haar measure m_G on G gives rise to a right invariant Haar measure μ_G on G by the formula

$$\forall B \in \mathcal{B}(G), \quad \mu_G(B) = m_G(B^{-1}).$$

The next proposition shows that any left invariant Haar measure has full support.

PROPOSITION 1.5. *Let G be any locally compact group and m_G any left invariant Haar measure on G. Then* $\mathrm{supp}(m_G) = G$. *Moreover, for every $f \in \mathrm{C}_c(G)_+$ such that $f \neq 0$, we have $\int_G f(h)\, \mathrm{d}m_G(h) > 0$.*

PROOF. Since $m_G \neq 0$, Conditions (ii) and (iii) in the definition of regularity imply that there exists a compact subset $K \subset G$ such that $0 < m_G(K) < +\infty$. Let $U \subset G$ be any nonempty open subset. There exist $g_1, \dots, g_n \in G$ such that $K \subset \bigcup_{i=1}^n g_i U$. This implies that

$$0 < m_G(K) \leq m_G(\bigcup_{i=1}^n g_i U) \leq \sum_{i=1}^n m_G(g_i U) = n \cdot m_G(U)$$

and so $m_G(U) > 0$. Thus, $\mathrm{supp}(m_G) = G$.

Moreover, let $f \in \mathrm{C}_c(G)_+$ such that $f \neq 0$. Then there exists $\varepsilon > 0$ and an open subset $U \subset G$ such that $f(h) \geq \varepsilon$ for every $h \in U$. This implies that

$$\int_G f(h)\, \mathrm{d}m_G(h) \geq \int_U \varepsilon\, \mathrm{d}m_G(h) = \varepsilon \cdot m_G(U) > 0.$$

This finishes the proof. □

The next proposition gives a characterization of compact groups in terms of the Haar measure.

PROPOSITION 1.6. *Let G be any locally compact group and m_G any left invariant Haar measure on G.*

Then G is compact if and only if $m_G(G) < +\infty$.

PROOF. Firstly, assume that G is compact. Then by regularity we have $m_G(G) < +\infty$.

Secondly, assume that G is not compact. Take a compact neighborhood $K \subset G$ of $e \in G$ and set $g_0 = e$. We have $m_G(K) > 0$ by Proposition 1.5. Since KK^{-1} is compact, there exists $g_1 \in G$ such that $g_1 \in G \setminus KK^{-1}$. This implies that $g_1 K \cap K = \emptyset$. By induction, define $g_n \in G$ so that

$g_n \in G \setminus (K \cup g_1 K \cup \cdots \cup g_{n-1}K)K^{-1}$. It follows that $(g_n K)_{n\in\mathbb{N}}$ are pairwise disjoint. This implies that

$$m_G(G) \geq m_G(\bigcup_{n\in\mathbb{N}} g_n K) = \sum_{n\in\mathbb{N}} m_G(g_n K) = +\infty \cdot m_G(K) = +\infty.$$

This finishes the proof. □

Let G be any locally compact group and m_G any left invariant Haar measure on G. The measure m_G need not be right invariant. For every $g \in G$, define the nonzero regular Borel measure m_G^g on G by the formula $m_G^g(B) = m_G(Bg)$ for every $B \in \mathcal{B}(G)$. Since m_G^g is a left invariant Haar measure, there exists a element $\Delta_G(g) \in \mathbb{R}_+^*$ such that $m_G^g = \Delta_G(g)\, m_G$. Then $\Delta_G : G \to \mathbb{R}_+^* : g \mapsto \Delta_G(g)$ is a group homomorphism and is called the *modular function* on G. The modular function Δ_G does not depend on the choice of the left invariant Haar measure m_G on G. Moreover, we have

$$\forall f \in \mathrm{C}_c(G), \forall g \in G, \int_G f(hg^{-1})\,\mathrm{d}m_G(h) = \Delta_G(g) \int_G f(h)\,\mathrm{d}m_G(h). \tag{1.1}$$

The left invariant Haar measure m_G is right invariant if and only if $\Delta_G \equiv 1$. In that case, we say that G is *unimodular*. We then simply refer to m_G as a Haar measure on G.

Proposition 1.7. *Let G be any locally compact group and m_G any left invariant Haar measure on G. Then the modular function $\Delta_G : G \to \mathbb{R}_+^*$ is continuous. Moreover, we have*

$$\forall f \in \mathrm{C}_c(G), \quad \int_G f(h^{-1})\,\mathrm{d}m_G(h) = \int_G \Delta_G(h^{-1}) f(h)\,\mathrm{d}m_G(h).$$

Proof. Choose $\varphi \in \mathrm{C}_c(G)$ such that $\kappa \doteq \int_G \varphi(h)\,\mathrm{d}m_G(h) \neq 0$. Set $Q = \mathrm{supp}(f)$. Then we have

$$\forall g \in G, \quad \Delta_G(g) = \frac{\int_G \varphi(hg^{-1})\,\mathrm{d}m_G(h)}{\int_G \varphi(h)\,\mathrm{d}m_G(h)}.$$

Choose a compact neighborhood $K \subset G$ of $e \in G$. Let $\varepsilon > 0$. Since φ is uniformly continuous by Lemma 1.8, there exists a neighborhood U of $e \in G$ such that $U \subset K$, $U^{-1} = U$ and

$$\forall u \in U, \quad \sup\left\{|\varphi(hu^{-1}) - \varphi(h)| \mid h \in G\right\} \leq \frac{\varepsilon\kappa}{m_G(QK)}.$$

Then for every $u \in U$, we have

$$\begin{aligned} |\Delta_G(u) - 1| &\leq \frac{1}{\kappa} \int_G |\varphi(hu^{-1}) - \varphi(h)|\,\mathrm{d}m_G(h) \\ &\leq \frac{1}{\kappa} m_G(QK) \frac{\varepsilon\kappa}{m_G(QK)} = \varepsilon. \end{aligned}$$

This implies that $\Delta_G : G \to \mathbb{R}_+^*$ is continuous at the identity element $e \in G$ and so Δ_G is continuous.

Next, observe that both of the positive linear functionals

$$\mathrm{C}_c(G) \to \mathbb{C} : f \mapsto \int_G f(h^{-1})\,\mathrm{d}m_G(h)$$
$$\mathrm{C}_c(G) \to \mathbb{C} : f \mapsto \int_G \Delta(h^{-1})f(h)\,\mathrm{d}m_G(h)$$

define a nonzero right invariant regular Borel measure on G. Thus, there exists $c > 0$ such that

$$\forall f \in \mathrm{C}_c(G), \quad \int_G f(h^{-1})\,\mathrm{d}m_G(h) = c\int_G \Delta_G(h^{-1})f(h)\,\mathrm{d}m_G(h)$$

Define $\widehat{\varphi} \in \mathrm{C}_c(G)$ by the formula $\widehat{\varphi}(h) = \varphi(h^{-1})$ for every $h \in G$. Then we have

$$\begin{aligned} 0 \neq \int_G \varphi(h)\,\mathrm{d}m_G(h) &= \int_G \widehat{\varphi}(h^{-1})\,\mathrm{d}m_G(h) \\ &= c\int_G \Delta_G(h^{-1})\widehat{\varphi}(h)\,\mathrm{d}m_G(h) \\ &= c\int_G \Delta_G(h^{-1})\varphi(h^{-1})\,\mathrm{d}m_G(h) \\ &= c^2\int_G \Delta_G(h^{-1})\Delta_G(h)\varphi(h)\,\mathrm{d}m_G(h) \\ &= c^2\int_G \varphi(h)\,\mathrm{d}m_G(h). \end{aligned}$$

This implies that $c = 1$. $\square$

In the proof of Proposition 1.7, we use the following technical result. Denote by $(\mathrm{C}_b(G), \|\cdot\|_\infty)$ the Banach space of all bounded continuous functions on G endowed with the supremum norm. Denote by $\lambda : G \curvearrowright \mathrm{C}_b(G)$ (resp. $\rho : G \curvearrowright \mathrm{C}_b(G)$) the left (resp. right) translation action defined by $(\lambda(g)f)(h) = f(g^{-1}h)$ (resp. $(\rho(g)f)(h) = f(hg)$) for all $g, h \in G$ and all $f \in \mathrm{C}_b(G)$.

Lemma 1.8. *Let G be any locally compact group and $f \in \mathrm{C}_c(G)$ any compactly supported continuous function. Then for every $\varepsilon > 0$, there exists a symmetric neighborhood $U \subset G$ of $e \in G$ such that*

$$\sup\{\|\lambda(u)f - f\|_\infty, \|\rho(u)f - f\|_\infty \mid u \in U\} < \varepsilon.$$

Then we say that $f \in \mathrm{C}_c(G)$ is uniformly continuous.

Proof. Let $f \in \mathrm{C}_c(G)$ and set $Q = \mathrm{supp}(f)$. Let $\varepsilon > 0$ and fix a symmetric compact neighborhood $V \subset G$ of $e \in G$. For every $g \in G$, there exists an open neighborhood $W_g \subset G$ of $g \in G$ such that for all $w_1, w_2 \in W_g$, we have $|f(w_1) - f(w_2)| < \varepsilon$. For every $g \in G$, choose an open symmetric neighborhood $U_g \subset G$ of $e \in G$ such that $gU_gU_g \cup U_gU_gg \subset W_g$. Then for every $g \in G$, $gU_g \cap U_gg$ is an open neighborhood of $g \in G$. Since VQV is compact, there exist $n \geq 1$ and $g_1, \dots, g_n \in G$ such that

$VQV \subset \bigcup_{i=1} g_i U_{g_i} \cap U_{g_i} g_i$. Define $U = V \cap \bigcap_{i=1}^n U_{g_i}$ which is a symmetric neighborhood of the identity $e \in G$. Then for every $u \in U$ and every $g \in G$, we consider the following situations:

- If $g \in VQV$, then there exists $1 \leq i \leq n$ such that $g \in g_i U_{g_i} \cap U_{g_i} g_i$. Since $u \in U \subset U_{g_i}$, we have $gu \in g_i U_{g_i} U_{g_i} \subset W_{g_i}$ and $ug \in U_{g_i} U_{g_i} g_i \subset W_{g_i}$. It follows that $|f(gu) - f(g)| < \varepsilon$ and $|f(gu) - f(g)| < \varepsilon$.
- If $g \notin VQV$, then $gu \notin Q$ and $ug \notin Q$. It follows that $f(g) = f(ug) = f(gu) = 0$.

We have showed that for every $u \in U$ and every $g \in G$, we have $|f(gu) - f(g)| < \varepsilon$ and $|f(gu) - f(g)| < \varepsilon$. □

Let (G, m_G, Δ_G) and (H, m_H, Δ_H) be any locally compact groups with their respective left invariant Haar measure and modular function. Let $\sigma : G \curvearrowright H$ be any continuous action by continuous group automorphisms and write $G \ltimes H$ for the locally compact semi-direct product group. Recall that the group law on $G \ltimes H$ is given by

$$\forall g_1, g_2 \in G, \forall h_1, h_2 \in H, \quad (g_1, h_1) \cdot (g_2, h_2) = (g_1 g_2, \sigma_{g_2}^{-1}(h_1) h_2).$$

The next proposition provides an explicit calculation of the Haar measure and the modular function on $G \ltimes H$.

Proposition 1.9. *The regular Borel measure $m_{G \ltimes H}$ defined on $G \ltimes H$ by the formulae*

$$\begin{aligned}
\forall f \in \mathrm{C}_c(G \ltimes H), \int_{G \ltimes H} f(g,h) \,\mathrm{d}m_{G \ltimes H}(h) \\
= \int_H \left(\int_G f(g,h) \,\mathrm{d}m_G(g) \right) \mathrm{d}m_H(h) \\
= \int_G \left(\int_H f(g,h) \,\mathrm{d}m_H(h) \right) \mathrm{d}m_G(g)
\end{aligned} \tag{1.2}$$

is a left invariant Haar measure on $G \ltimes H$. Moreover, the modular function $\Delta_{G \ltimes H} : G \ltimes H \to \mathbb{R}_+^*$ satisfies

$$\forall (g,h) \in G \ltimes H, \quad \Delta_{G \ltimes H}(g,h) = \rho(g)\, \Delta_G(g)\, \Delta_H(h)$$

where $\rho : G \to \mathbb{R}_+^*$ is the continuous function defined by the formula

$$\forall f \in \mathrm{C}_c(H), \forall g \in G, \quad \int_H f(\sigma_g(h)) \,\mathrm{d}m_H(h) = \rho(g) \int_H f(h) \,\mathrm{d}m_H(h).$$

Proof. Fubini's theorem implies that for every $f \in \mathrm{C}_c(G \ltimes H)$, we have

$$\int_H \left(\int_G f(g,h) \,\mathrm{d}m_G(g) \right) \mathrm{d}m_H(h) = \int_G \left(\int_H f(g,h) \,\mathrm{d}m_H(h) \right) \mathrm{d}m_G(g).$$

Denote by $m_{G\ltimes H}$ the unique regular Borel measure on $G \ltimes H$ defined by (1.2). For every $f \in \mathrm{C}_c(G \ltimes H)$ and every $(g_1, h_1) \in G \ltimes H$, we have

$$\begin{aligned}
&\int_{G\ltimes H} f((g_1,h_1)\cdot(g_2,h_2))\,\mathrm{d}m_{G\ltimes H}(g_2,h_2)\\
&\qquad = \int_{G\ltimes H} f(g_1g_2, \sigma_{g_2}^{-1}(h_1)h_2)\,\mathrm{d}m_{G\ltimes H}(g_2,h_2)\\
&\qquad = \int_G \left(\int_H f(g_1g_2,h_2)\,\mathrm{d}m_H(h_2)\right)\mathrm{d}m_G(g_2)\\
&\qquad = \int_H \left(\int_G f(g_2,h_2)\,\mathrm{d}m_G(g_2)\right)\mathrm{d}m_H(h_2)\\
&\qquad = \int_{G\ltimes H} f(g_2,h_2)\,\mathrm{d}m_{G\ltimes H}(g_2,h_2).
\end{aligned}$$

This shows that $m_{G\ltimes H}$ is a left invariant Haar measure on $G \ltimes H$.

Consider the function $\rho : G \to \mathbb{R}_+^*$ as defined above. For every $f \in \mathrm{C}_c(G \ltimes H)$ and every $(g_2, h_2) \in G \ltimes H$, we have

$$\begin{aligned}
&\int_{G\ltimes H} f((g_1,h_1)\cdot(g_2,h_2)^{-1})\,\mathrm{d}m_{G\ltimes H}(g_1,h_1)\\
&\qquad = \int_{G\ltimes H} f(g_1g_2^{-1}, \sigma_{g_2}(h_1h_2^{-1}))\,\mathrm{d}m_{G\ltimes H}(g_1,h_1)\\
&\qquad = \Delta_H(h_2)\int_G \left(\int_H f(g_1g_2^{-1},\sigma_{g_2}(h_1))\,\mathrm{d}m_H(h_1)\right)\mathrm{d}m_G(g_1)\\
&\qquad = \rho(g_2)\,\Delta_H(h_2)\int_G \left(\int_H f(g_1g_2^{-1},h_1)\,\mathrm{d}m_H(h_1)\right)\mathrm{d}m_G(g_1)\\
&\qquad = \rho(g_2)\,\Delta_G(g_2)\,\Delta_H(h_2)\int_H \left(\int_G f(g_1,h_1)\,\mathrm{d}m_G(g_1)\right)\mathrm{d}m_H(h_1)\\
&\qquad = \rho(g_2)\,\Delta_G(g_2)\,\Delta_H(h_2)\int_{G\ltimes H} f(g_1,h_1)\,\mathrm{d}m_{G\ltimes H}(g_1,h_1)
\end{aligned}$$

and hence $\Delta_{G\ltimes H}(g_2, h_2) = \rho(g_2)\,\Delta_G(g_2)\,\Delta_H(h_2)$. □

Examples 1.10. Here are some examples of unimodular locally compact groups. Let $d \geq 1$.

(i) Any group G endowed with the discrete topology is unimodular. Indeed, in that case the counting measure m_G is a nonzero regular Borel measure on G that is clearly both left and right invariant.

(ii) Any compact group G is unimodular. Indeed, fix a left invariant Haar measure m_G on G. Then $\Delta_G(G) < \mathbb{R}_+^*$ is a compact subgroup and so $\Delta_G(G) = \{1\}$. This shows that $\Delta_G \equiv 1$ and so G is unimodular.

(iii) Any abelian locally compact group G is unimodular. The Lebesgue measure $\mathrm{d}x_1 \cdots \mathrm{d}x_d$ on $\mathbb{R}^d$ is a Haar measure.

(iv) Recall that the general linear group $\mathrm{GL}_d(\mathbb{R})$ can be regarded as the open (dense) subset of invertible matrices in $\mathbb{M}_d(\mathbb{R}) \cong \mathbb{R}^d \times \cdots \times \mathbb{R}^d$. For every $g \in \mathrm{GL}_d(\mathbb{R})$, the Jacobian of the diffeomorphism

$$L_g : \mathbb{M}_d(\mathbb{R}) \to \mathbb{M}_d(\mathbb{R}) : (x_1, \dots, x_d) \mapsto (gx_1, \dots, gx_d)$$

is equal to $\det(g)^d$. It follows that a left invariant Haar measure m_G on $G = \mathrm{GL}_d(\mathbb{R})$ is given by

$$\mathrm{d}m_G(g) = \frac{1}{\det(g)^d} \prod_{1 \leq i,j \leq d} \mathrm{d}g_{ij}, \quad g = (g_{ij})_{ij}.$$

For every $g \in \mathrm{GL}_d(\mathbb{R})$, since the Jacobian of the diffeomorphism

$$R_g : \mathbb{M}_d(\mathbb{R}) \to \mathbb{M}_d(\mathbb{R}) : x \mapsto xg$$

is also equal to $\det(g)^d$, it follows that m_G is right invariant and so $G = \mathrm{GL}_d(\mathbb{R})$ is unimodular.

(v) Recall that the special linear group $\mathrm{SL}_d(\mathbb{R}) < \mathrm{GL}_d(\mathbb{R})$ is defined by $\mathrm{SL}_d(\mathbb{R}) = \ker(\det)$. It follows from Iwasawa's theorem that the only normal subgroups of $\mathrm{SL}_d(\mathbb{R})$ are $\{1\}$, $\{\pm 1\}$ and $\mathrm{SL}_d(\mathbb{R})$. This implies that $\ker(\Delta_{\mathrm{SL}_d(\mathbb{R})}) = \mathrm{SL}_d(\mathbb{R})$ and so $\mathrm{SL}_d(\mathbb{R})$ is unimodular.

(vi) For every $d \geq 2$, the strict upper triangular subgroup $G = \mathrm{T}_d(\mathbb{R})$ defined as the group of all matrices $g = (g_{ij})_{ij}$ such that $g_{ij} = 0$ for all $1 \leq j < i \leq d$ and $g_{ii} = 1$ for all $1 \leq i \leq d$ is homeomorphic with $\mathbb{R}^{\frac{d(d-1)}{2}}$. Under this identification, the Lebesgue measure on $\mathbb{R}^{\frac{d(d-1)}{2}}$ gives rise to a left and right invariant Haar measure m_G on G defined as

$$\mathrm{d}m_G(n) = \prod_{1 \leq i < j \leq d} \mathrm{d}n_{ij}, \quad n = (n_{ij})_{ij}.$$

Indeed, for all $i > j$ and all $g, n \in \mathrm{T}_d(\mathbb{R})$, we have $(gn)_{ij} = g_{ij} + n_{ij} + \sum_{j<k<i} g_{ik} n_{kj}$. Endow the set $\{(i,j) \mid 1 \leq j < i \leq d\}$ with the lexicographical order. Then for every $g \in \mathrm{T}_d(\mathbb{R})$, the Jacobian matrix of the diffeomorphism $\mathrm{T}_d(\mathbb{R}) \to \mathrm{T}_d(\mathbb{R}) : n \mapsto gn$ is upper triangular with diagonal entries all equal to 1. This implies that the Jacobian of the diffeomorphism $\mathrm{T}_d(\mathbb{R}) \to \mathrm{T}_d(\mathbb{R}) : n \mapsto gn$ is equal to 1. The same argument shows that for every $g \in \mathrm{T}_d(\mathbb{R})$, the Jacobian of the diffeomorphism $\mathrm{T}_d(\mathbb{R}) \to \mathrm{T}_d(\mathbb{R}) : n \mapsto ng$ is equal to 1. Thus, $G = \mathrm{T}_d(\mathbb{R})$ is unimodular.

2. Lattices in locally compact groups

Let G be any locally compact group and $\Gamma < G$ any discrete subgroup. We say that a Borel subset $\mathcal{F} \subset G$ is a *Borel fundamental domain* (for the right translation action $\Gamma \curvearrowright G$) if

$$\forall \gamma_1, \gamma_2 \in \Gamma,\ \gamma_1 \neq \gamma_2 \ \Rightarrow\ \mathcal{F}\gamma_1 \cap \mathcal{F}\gamma_2 = \emptyset \quad \text{and} \quad \bigcup_{\gamma \in \Gamma} \mathcal{F}\gamma = G.$$

Denote by $G/\Gamma = \{g\Gamma \mid g \in G\}$ the quotient space and by $p : G \to G/\Gamma : g \mapsto g\Gamma$ the quotient map. Endow G/Γ with the quotient topology.

PROPOSITION 1.11. *Keep the same notation as above. The following assertions hold:*

(i) *The quotient map $p : G \to G/\Gamma$ is continuous and open and G/Γ is Hausdorff and locally compact. Moreover, the action map $G \times G/\Gamma \to G/\Gamma : (g, x) \mapsto gx$ is continuous.*
(ii) *If G/Γ is compact, then there exists a Borel fundamental domain $\mathcal{F} \subset G$ that is relatively compact in G.*
(iii) *If G is second countable, then G/Γ is second countable. Moreover, there exists a Borel fundamental domain $\mathcal{F} \subset G$ such that for every compact subset $Y \subset G/\Gamma$, the subset $p^{-1}(Y) \cap \mathcal{F} \subset G$ is relatively compact in G.*

PROOF. (i) Endow the quotient space $G/\Gamma = \{g\Gamma \mid g \in G\}$ with the quotient topology. By definition, a subset $V \subset G/\Gamma$ is open if and only if $p^{-1}(V) \subset G$ is open. Then the quotient topology is the finest topology on G/Γ that makes the quotient map $p : G \to G/\Gamma$ continuous. Let now $U \subset G$ be any open set. Then $p^{-1}(p(U)) = p^{-1}(\{h\Gamma \mid h \in U\}) = \bigcup_{\gamma \in \Gamma} U\gamma$ is open and so is $p(U) \subset G/\Gamma$ is open. This shows that $p : G \to G/\Gamma$ is open.

Let $x_1, x_2 \in G/\Gamma$ with $x_1 \neq x_2$. Write $x_1 = g_1\Gamma$ and $x_2 = g_2\Gamma$. Note that $g_2 \notin g_1\Gamma$. Choose a compact neighborhood $U_1 \subset G$ (resp. $U_2 \subset G_2$) of $g_1 \in G$ (resp. $g_2 \in G$). Since $U_2^{-1}U_1 \subset G$ is compact and since $\Gamma < G$ is discrete, the set $\Lambda \doteq \{\gamma \in \Gamma \mid U_1 \cap U_2\gamma \neq \emptyset\}$ is finite. For every $\gamma \in \Lambda$, since $g_1 \neq g_2\gamma$, there exist neighborhoods U_γ of $g_1 \in G$ and V_γ of $g_2\gamma \in G$ such that $U_\gamma \cap V_\gamma = \emptyset$. Set

$$\overline{U}_1 = U_1 \cap \bigcap_{\gamma \in \Lambda} U_\gamma \quad \text{and} \quad \overline{U}_2 = U_2 \cap \bigcap_{\gamma \in \Lambda} V_\gamma\gamma^{-1}.$$

Then for every $\gamma \in \Gamma$, we have $\overline{U}_1 \cap \overline{U}_2\gamma = \emptyset$. Indeed, if $\gamma \in \Gamma \setminus \Lambda$, then $U_1 \cap U_2\gamma = \emptyset$. If $\gamma \in \Lambda$, then $U_\gamma \cap (V_\gamma\gamma^{-1})\gamma = \emptyset$. Thus, we have $p(\overline{U}_1) \cap p(\overline{U}_2) = \emptyset$. This shows that G/Γ is Hausdorff.

Let $x = g\Gamma \in G/\Gamma$ be any element. Choose a compact neighborhood $K \subset G$ of $e \in G$. Then gK is a compact neighborhood of $g \in G$ and so $p(gK)$ is a compact neighborhood of $x \in G/\Gamma$. This shows that G/Γ is locally compact.

Define the action map $a : G \times G/\Gamma \to G/\Gamma : (g, x) \mapsto gx$. Recall that the multiplication map $m : G \times G \to G$ is continuous. Since the map $\mathrm{id}_G \times p : G \times G \to G \times G/\Gamma : (g, h) \mapsto (g, h\Gamma)$ is continuous and open, the commutative diagram

$$\begin{array}{ccc} G \times G & \xrightarrow{m} & G \\ \Big\downarrow{\scriptstyle \mathrm{id} \times p} & & \Big\downarrow{\scriptstyle p} \\ G \times G/\Gamma & \xrightarrow{a} & G/\Gamma \end{array}$$

shows that the action map $a : G \times G/\Gamma \to G/\Gamma$ is continuous.

(ii) Since $\Gamma < G$ is discrete, there exists an open neighborhood $V \subset G$ of $e \in G$ such that $V \cap \Gamma = \{e\}$. Since the map $G \times G \to G : (g, h) \mapsto g^{-1}h$ is continuous, there exists an open neighborhood $U \subset G$ of $e \in G$ such that $U^{-1}U \subset V$. Replacing U with $U \cap K$ where K is a relatively compact open neighborhood of $e \in G$, we may assume that $U \subset G$ is relatively compact. Since G/Γ is compact and since $(p(gU)_{g\in G})$ is an open covering of G/Γ, there exist $g_1, \dots, g_n \in G$ such that $G/\Gamma = \bigcup_{i=1}^n p(g_iU)$. Define the Borel subset

$$\mathcal{F} = \bigcup_{i=1}^{n} \left(g_iU \setminus \bigcup_{j<i} g_jU\Gamma \right).$$

By construction, $\mathcal{F} \subset G$ is relatively compact. Then we have $\bigcup_{\gamma\in\Gamma} \mathcal{F}\gamma = \bigcup_{i=1}^n g_iU\Gamma = p^{-1}(\bigcup_{i=1}^n p(g_iU)) = p^{-1}(G/\Gamma) = G$. Let $\gamma_1, \gamma_2 \in \Gamma$ be any elements such that $\mathcal{F}\gamma_1 \cap \mathcal{F}\gamma_2 \neq \emptyset$. Up to exchanging γ_1 and γ_2, there exist $i \geq j$ and $u_1, u_2 \in U$ such that $g_iu_1\gamma_1 = g_ju_2\gamma_2$. By construction and since $g_iu_1 = g_ju_2\gamma_2\gamma_1^{-1} \in g_iU \cap g_jU\Gamma$, we necessarily have $i = j$. Then $u_1\gamma_1 = u_2\gamma_2$ and so $u_2^{-1}u_1 = \gamma_2\gamma_1^{-1} \in U^{-1}U \cap \Gamma \subset V \cap \Gamma = \{e\}$. This shows that $\gamma_1 = \gamma_2$ and thus $\mathcal{F} \subset G$ is a Borel fundamental domain.

(iii) Choose a countable basis $(U_n)_{n\in\mathbb{N}}$ for the topology on G. Let $V \subset G/\Gamma$ be any open set. Then $p^{-1}(V) = \bigcup_{\gamma\in\Gamma} V\gamma \subset G$ is open and so there exists a subfamily $(U_{n_k})_k$ such that $p^{-1}(V) = \bigcup_k U_{n_k}$. Then we have $V = p(p^{-1}(V)) = \bigcup_k p(U_{n_k})$. This shows that $(p(U_n))_{n\in\mathbb{N}}$ is a countable basis for the quotient topology on G/Γ and so G/Γ is second countable. For every $n \in \mathbb{N}$, choose $g_n \in U_n$.

As before, there exist open neighborhoods $U, V \subset G$ of $e \in G$ such that $U \subset G$ is relatively compact, $U^{-1}U \subset V$ and $V \cap \Gamma = \{e\}$. We claim that $G = \bigcup_{n\in\mathbb{N}} g_nU$. Indeed, for every $g \in G$, $gU^{-1} \subset G$ is an open set and hence there exists $n \in \mathbb{N}$ such that $U_n \subset gU^{-1}$. This implies that there exists $u \in U$ such that $g_n = gu^{-1}$ or equivalently $g = g_nu$ and thus $g \in g_nU$. Define the Borel subset

$$\mathcal{F} = \bigcup_{n\in\mathbb{N}} \left(g_nU \setminus \bigcup_{k<n} g_kU\Gamma \right).$$

Then we have $\bigcup_{\gamma\in\Gamma} \mathcal{F}\gamma = \bigcup_{n\in\mathbb{N}} g_nU\Gamma = G$. Let $\gamma_1, \gamma_2 \in \Gamma$ be any elements such that $\mathcal{F}\gamma_1 \cap \mathcal{F}\gamma_2 \neq \emptyset$. Up to exchanging γ_1 and γ_2, there exist $m \geq n$ and $u_1, u_2 \in U$ such that $g_mu_1\gamma_1 = g_nu_2\gamma_2$. By construction and since $g_mu_2 = g_nu_2\gamma_2\gamma_1^{-1} \in g_mU \cap g_nU\Gamma$, we necessarily have $m = n$. Then $u_1\gamma_1 = u_2\gamma_2$ and so $u_2^{-1}u_1 = \gamma_2\gamma_1^{-1} \in U^{-1}U \cap \Gamma \subset V \cap \Gamma = \{e\}$. This shows that $\gamma_1 = \gamma_2$ and thus $\mathcal{F} \subset G$ is a Borel fundamental domain. Let $Y \subset G/\Gamma$ be any compact subset. Since $(p(g_nU))_{n\in\mathbb{N}}$ is an open covering of Y, there exist $n_1, \dots, n_k \in \mathbb{N}$ such that $Y \subset \bigcup_{i=1}^k p(g_{n_i}U)$. Then we have $p^{-1}(Y) \cap \mathcal{F} \subset \bigcup_{j=0}^{n_k}(g_jU \setminus \bigcup_{i<j} g_iU\Gamma)$ and so $p^{-1}(Y) \cap \mathcal{F} \subset G$ is relatively compact. $\square$

Observe that when G is a locally compact σ-compact group, any discrete subgroup $\Gamma < G$ is necessarily countable. Indeed, since G is σ-compact, the left invariant Haar measure m_G is σ-finite. We may then choose a Borel probability measure $\mu \in \mathrm{Prob}(G)$ such that $\mu \sim m_G$. We may also choose open neighborhoods $U, V \subset G$ of $e \in G$ such that $UU^{-1} \subset V$ and $V \cap \Gamma = \{e\}$. Then $(\gamma U)_{\gamma \in \Gamma}$ is a family of pairwise disjoint open subsets. Moreover, since $m_G(\gamma U) = m_G(U) > 0$ for every $\gamma \in \Gamma$, it follows that $\mu(\gamma U) > 0$ for every $\gamma \in \Gamma$. This implies that Γ is necessarily countable.

COROLLARY 1.12. *Let G be any locally compact second countable group and $\Gamma < G$ any discrete subgroup. Then there exists a Borel map $\sigma : G/\Gamma \to G$ such that*

- $\sigma(G/\Gamma) = \mathcal{F}$ *is a Borel fundamental domain,*
- $\sigma(\Gamma) = e$,
- $x = \sigma(x)\Gamma$ *for every* $x \in G/\Gamma$,
- $\sigma(Y) \subset G$ *is relatively compact for every compact subset* $Y \subset G/\Gamma$.

We then simply say that $\sigma : G/\Gamma \to G$ is a Borel section.

PROOF. Choose a Borel fundamental domain $\mathcal{F} \subset G$ as in Proposition 1.11(iii) such that $e \in \mathcal{F}$. Then $p|_{\mathcal{F}} : \mathcal{F} \to G/\Gamma$ is Borel and bijective. This implies that the map $\sigma = (p|_{\mathcal{F}})^{-1} : G/\Gamma \to G$ is Borel (see [**Zi84**, Theorem A.4]) and satisfies all the required properties. $\square$

DEFINITION 1.13. Let G be any locally compact group and $\Gamma < G$ any discrete subgroup. We say that $\Gamma < G$ is *uniform* or *cocompact* if G/Γ is compact.

We say that $\Gamma < G$ is a *lattice* if there exists a G-invariant regular Borel probability measure $\nu \in \mathrm{Prob}(G/\Gamma)$.

Define the linear mapping $\mathcal{T} : \mathrm{C}_c(G) \to \mathrm{C}_c(G/\Gamma) : f \mapsto \overline{f}$ by the formula

$$\forall g \in G, \quad \overline{f}(g\Gamma) = \sum_{\gamma \in \Gamma} f(g\gamma).$$

We claim that $\mathcal{T} : \mathrm{C}_c(G) \to \mathrm{C}_c(G/\Gamma)$ is surjective. Indeed, let $\varphi \in \mathrm{C}_c(G/\Gamma)$ be any function and denote by $Q = \mathrm{supp}(\varphi) \subset G/\Gamma$ its compact support. Choose a relatively compact open neighborhood $V \subset G$ of $e \in G$. Then there exist $g_1, \dots, g_n \in G$ such that $Q \subset \bigcup_{i=1}^n p(g_i V)$. Set $K = p^{-1}(Q) \cap \bigcup_{i=1}^n g_i \overline{V}$. Then $K \subset G$ is a compact subset such that $p(K) = Q$. By Urysohn's lemma (see e.g. [**DE14**, Lemma A.8.1(ii)]), we may choose $f_K \in \mathrm{C}_c(G)_+$ such that $f|_K \equiv 1_K$.

Define the function $f : G \to \mathbb{C}$ by the formula $f(g) = \frac{\varphi(g\Gamma)}{\mathcal{T}(f_K)(g\Gamma)} f_K(g)$ if $\mathcal{T}(f_K)(g\Gamma) \neq 0$ and $f(g) = 0$ otherwise. Then $\mathrm{supp}(f) \subset \mathrm{supp}(f_K)$ is compact and f is continuous on G since $\mathcal{T}(f_K)(g\Gamma) > 0$ on a neighborhood of Q. Thus, $f \in \mathrm{C}_c(G)$ and we have $\mathcal{T}(f) = \varphi$.

PROPOSITION 1.14. *Let G be any locally compact group and $\Gamma < G$ any uniform discrete subgroup. Then G is unimodular and $\Gamma < G$ is a lattice.*

If G is moreover compactly generated, then $\Gamma < G$ is finitely generated.

PROOF. Fix a right invariant Haar measure μ_G on G. Consider the positive linear functional

$$\Phi : \mathrm{C}_c(G/\Gamma) \to \mathbb{C} : \overline{f} \mapsto \int_G f(g)\, \mathrm{d}\mu_G(g).$$

In order to check that Φ is well-defined, it suffices to show that if $\varphi \in \mathrm{C}_c(G)$ is such that $\overline{\varphi} = 0$, then we have $\int_G \varphi(g)\, \mathrm{d}\mu_G(g) = 0$. Indeed, for every $\psi \in \mathrm{C}_c(G)$, using Fubini's theorem, we have

$$\begin{aligned}\int_G \overline{\varphi}(h\Gamma)\psi(h)\, \mathrm{d}\mu_G(h) &= \sum_{\gamma\in\Gamma}\int_G \varphi(h\gamma)\psi(h)\, \mathrm{d}\mu_G(h) \\ &= \sum_{\gamma\in\Gamma}\int_G \varphi(h)\psi(h\gamma^{-1})\, \mathrm{d}\mu_G(h) \\ &= \int_G \varphi(h)\overline{\psi}(h\Gamma)\, \mathrm{d}\mu_G(h).\end{aligned}$$

Since the map $\mathrm{C}_c(G) \to \mathrm{C}_c(G/\Gamma) : f \mapsto \overline{f}$ is surjective, there exists $\psi \in \mathrm{C}_c(G)$ such that $\overline{\psi} \equiv 1$ on the compact subset $\mathrm{supp}(\varphi)\Gamma \subset G/\Gamma$. Therefore, we obtain

$$\int_G \varphi(h)\, \mathrm{d}\mu_G(h) = \int_G \varphi(h)\overline{\psi}(h\Gamma)\, \mathrm{d}\mu_G(h) = \int_G \overline{\varphi}(h\Gamma)\psi(h)\, \mathrm{d}\mu_G(h) = 0.$$

By Riesz's representation theorem, there exists a unique regular Borel measure ν on G/Γ such that

$$\forall f \in \mathrm{C}_c(G), \quad \int_G f(h)\, \mathrm{d}\mu_G(h) = \int_G \overline{f}(h\Gamma)\, \mathrm{d}\nu(h\Gamma).$$

Note that the above argument does not use the fact that $\Gamma < G$ is uniform.

However, since $\Gamma < G$ is uniform, G/Γ is compact and we have $0 < \nu(G/\Gamma) < +\infty$. Up to normalization, we may assume that $\nu(G/\Gamma) = 1$.

Define the left invariant Haar measure m_G on G by the formula $m_G(B) = \mu_G(B^{-1})$ for every $B \in \mathcal{B}(G)$. Then for every $B \in \mathcal{B}(G)$ and every $g \in G$, we have

$$(g_*\mu_G)(B) = \mu_G(g^{-1}B) = m_G(B^{-1}g) = \Delta_G(g)\, m_G(B^{-1}) = \Delta_G(g)\, \mu_G(B)$$

and so $g_*\mu_G = \Delta_G(g)\, \mu_G$. By uniqueness in the previous construction, we obtain $g_*\nu = \Delta_G(g)\, \nu$ for every $g \in G$. Since $\nu \in \mathrm{Prob}(G/\Gamma)$ is a probability measure, we obtain $\Delta_G(g) = 1$ and $g_*\nu = \nu$ for every $g \in G$. Thus, $\Delta_G \equiv 1$ and so G is unimodular. Moreover, $\nu \in \mathrm{Prob}(G/\Gamma)$ is G-invariant and so $\Gamma < G$ is a lattice.

Assume moreover that G is compactly generated. Choose a compact subset $Q \subset G$ such that $e \in Q$ and $G = \bigcup_{n\geq 1} Q^n$. Since G/Γ is compact, we may choose a compact subset $K \subset G$ such that $p(K) = G/\Gamma$ (see the proof of surjectivity of the map $\mathcal{T} : \mathrm{C}_c(G) \to \mathrm{C}_c(G/\Gamma)$). Up to replacing Q by $Q \cup K$, we may further assume that $Q \cdot \Gamma = G$. Then $S_0 = Q \cap \Gamma$ is finite. Moreover, since Q^2 is compact, there exists a finite subset $S_1 \subset \Gamma$ such that $Q^2 \subset QS_1$. Indeed, otherwise we could find sequences $(g_n)_{n\in\mathbb{N}}$ in Q^2, $(h_n)_{n\in\mathbb{N}}$ in Q and $(\gamma_n)_{n\in\mathbb{N}}$ in Γ such that $g_n = h_n\gamma_n$ for every $n \in \mathbb{N}$ and $(\gamma_n)_{n\in\mathbb{N}}$ are pairwise distinct. This would imply that $\gamma_n = h_n^{-1}g_n \in Q^3 \cap \Gamma$ for every $n \in \mathbb{N}$. Since Q^3 is compact and $\Gamma < G$ is discrete, $Q^3 \cap \Gamma$ must be finite, a contradiction. Set $S = S_0 \cup S_1 \subset \Gamma$. Then $Q \cap \Gamma \subset S$ and for every $n \geq 1$, we have $Q^{n+1} \subset QS^n$. We claim that S is a finite generating set for Γ. Indeed, by construction, we have $Q \cap \Gamma \subset S$. Next, let $n \geq 1$ and $\gamma \in Q^{n+1} \cap \Gamma \subset QS^n \cap \Gamma$. Then $\gamma = g\gamma_n$ where $g \in Q$ and $\gamma_n \in S^n$. This implies that $\gamma\gamma_n^{-1} = g \in Q \cap \Gamma \subset S$. Then $\gamma = g\gamma_n \in SS^n = S^{n+1}$ and hence $Q^{n+1} \cap \Gamma \subset S^{n+1}$. This implies that $\Gamma = \bigcup_{n\geq 1} Q^n \cap \Gamma \subset \bigcup_{n\geq 1} S^n$ and so Γ is finitely generated. $\square$

PROPOSITION 1.15. *Let G be any locally compact group that possesses a lattice $\Gamma < G$. Then G is unimodular. Moreover, there is a unique G-invariant regular Borel probability measure $\nu \in \mathrm{Prob}(G/\Gamma)$.*

PROOF. Let $\nu \in \mathrm{Prob}(G/\Gamma)$ be a G-invariant regular Borel probability measure. We claim that there exists a unique left invariant Haar measure m_G on G such that

$$\forall f \in \mathrm{C}_c(G), \quad \int_G f(h)\,\mathrm{d}m_G(h) = \int_{G/\Gamma} \overline{f}(g\Gamma)\,\mathrm{d}\nu(g\Gamma). \tag{1.3}$$

Indeed, the well-defined positive linear functional

$$\mathrm{C}_c(G) \to \mathbb{C} : f \mapsto \int_{G/\Gamma} \overline{f}(g\Gamma)\,\mathrm{d}\nu(g\Gamma)$$

is left invariant. By Riesz's representation theorem, there exists a unique left invariant Haar measure m_G on G for which (1.3) holds.

Applying (1.1), for every $f \in \mathrm{C}_c(G)$ and every $\gamma \in \Gamma$, letting $f_\gamma \doteq f(\,\cdot\,\gamma^{-1}) \in \mathrm{C}_c(G)$, we have

$$\begin{aligned}\Delta_G(\gamma)\int_G f(h)\,\mathrm{d}m_G(h) &= \int_G f_\gamma(h)\,\mathrm{d}m_G(h)\\ &= \int_{G/\Gamma} \overline{f_\gamma}(h\Gamma)\,\mathrm{d}\nu(h\Gamma)\\ &= \int_{G/\Gamma} \overline{f}(h\Gamma)\,\mathrm{d}\nu(h\Gamma)\\ &= \int_G f(h)\,\mathrm{d}m_G(h).\end{aligned}$$

This implies that $\Delta_G(\gamma) = 1$ for every $\gamma \in \Gamma$. Consider the well-defined continuous mapping $\overline{\Delta} : G/\Gamma \to \mathbb{R}_+^* : g\Gamma \mapsto \Delta_G(g)$. Then $\eta = \overline{\Delta}_*\nu \in \mathrm{Prob}(\mathbb{R}_+^*)$ is a Borel probability measure that is invariant under the subgroup $\Delta_G(G) < \mathbb{R}_+^*$. Thus, we necessarily have $\Delta_G(G) = \{1\}$. This implies that $\Delta_G \equiv 1$ and so G is unimodular.

Observe that (1.3) implies that there is a unique G-invariant regular Borel probability measure $\nu \in \mathrm{Prob}(G/\Gamma)$. $\square$

The next proposition provides a group-theoretic characterization of uniform lattices in locally compact groups.

Proposition 1.16. *Let G be any locally compact group and $\Gamma < G$ any lattice. The following assertions are equivalent:*

(i) *$\Gamma < G$ is uniform.*
(ii) *There exists a compact neighborhood $U \subset G$ of $e \in G$ such that for every $g \in G$, we have $g\Gamma g^{-1} \cap U = \{e\}$.*

Proof. (i) $\Rightarrow$ (ii) Assume that $\Gamma < G$ is uniform. Since $\Gamma < G$ is discrete, we may choose a compact neighborhood $W \subset G$ of $e \in G$ such that $\Gamma \cap W = \{e\}$. Next, we may choose a symmetric compact neighborhood $V \subset W$ of $e \in G$ such that $VVV \subset W$. Observe that for every $h \in V$, we have

$$h\Gamma h^{-1} \cap V \subset h(\Gamma \cap h^{-1}Vh)h^{-1} \subset h(\Gamma \cap W)h^{-1} = \{e\}.$$

By compactness of G/Γ, there exist $n \geq 1$ and $g_1, \dots, g_n \in G$ such that $G/\Gamma = \bigcup_{i=1}^n g_i p(V)$. Set $U \doteq \bigcap_{i=1}^n g_i V g_i^{-1}$. Then for every $g \in G$, there exist $1 \leq i \leq n$ and $h \in V$ such that $g\Gamma = g_i h\Gamma$ and hence

$$g\Gamma g^{-1} \cap U = g_i h\Gamma h^{-1} g_i^{-1} \cap U \subset g_i(h\Gamma h^{-1} \cap V)g_i^{-1} = \{e\}.$$

(ii) $\Rightarrow$ (i) Denote by $\nu \in \mathrm{Prob}(G/\Gamma)$ the unique G-invariant regular Borel probability measure and by m_G the unique Haar measure on G such that (1.3) holds. Assume that there exists such a compact neighborhood $U \subset G$ of $e \in G$. Choose a compact neighborhood $V \subset G$ of $e \in G$ such that $V^{-1}V \subset U$. Choose a nonnegative function $\varphi \in \mathrm{C}_c(G)$ such that $0 \leq \varphi \leq 1$ and $\mathrm{supp}(\varphi) \subset V$. Set $\varepsilon \doteq \int_G \varphi(h)\, \mathrm{d}m_G(h)$.

For every $g \in G$, define $\varphi_g \doteq \varphi(\,\cdot\, g^{-1}) \in \mathrm{C}_c(G)$. Note that $0 \leq \varphi_g \leq 1$ and $\mathrm{supp}(\varphi_g) \subset Vg$. Moreover, we have $\mathrm{supp}(\overline{\varphi_g}) \subset Vg\Gamma$. Since m_G is right invariant, we have

$$\begin{aligned}
\varepsilon &= \int_G \varphi(h)\, \mathrm{d}m_G(h) \\
&= \int_G \varphi_g(h)\, \mathrm{d}m_G(h) \\
&= \int_{G/\Gamma} \overline{\varphi_g}(h\Gamma)\, \mathrm{d}\nu(h\Gamma) \\
&= \int_{Vg\Gamma} \overline{\varphi_g}(h\Gamma)\, \mathrm{d}\nu(h\Gamma) \\
&= \int_{Vg\Gamma} \sum_{\gamma \in \Gamma} \varphi_g(h\gamma)\, \mathrm{d}\nu(h\Gamma).
\end{aligned}$$

We claim that for every $h \in Vg\Gamma$, there is at most one $\gamma \in \Gamma$ such that $h\gamma \in Vg$. Indeed, if $\gamma_1, \gamma_2 \in \Gamma$ are elements such that $h\gamma_1, h\gamma_2 \in Vg$, then

$g\gamma_1^{-1}\gamma_2 g^{-1} \in V^{-1}V \subset U$. Since $g\Gamma g^{-1} \cap U = \{e\}$, we have $\gamma_1 = \gamma_2$. Since $0 \leq \varphi_g \leq 1$ and $\operatorname{supp}(\varphi_g) \subset V_g$, it follows that

$$\varepsilon = \int_{Vg\Gamma} \sum_{\gamma \in \Gamma} \varphi_g(h\gamma)\, \mathrm{d}\nu(h\Gamma) \leq \int_{Vg\Gamma} 1 \, \mathrm{d}\nu(h\Gamma) = \nu(Vg\Gamma).$$

We have showed that $\nu(Vg\Gamma) \geq \varepsilon$ for every $g \in G$.

Let $F \subset G$ be any finite subset for which for every $g, h \in F$ such that $g \neq h$, we have $Vg\Gamma \cap Vh\Gamma = \emptyset$. Then we have

$$\sharp F \cdot \varepsilon \leq \sum_{g \in F} \nu(Vg\Gamma) = \nu(\bigcup_{g \in F} Vg\Gamma) \leq 1$$

and hence $\sharp F \leq \varepsilon^{-1}$. We may then choose a maximal finite subset $F \subset G$ with the aforementioned property. It follows that for every $g \in G$, we have $Vg\Gamma \cap VF\Gamma \neq \emptyset$ and hence $g\Gamma \in V^{-1}VF\Gamma \subset UF\Gamma$. Since $UF\Gamma \subset G/\Gamma$ is compact, it follows that $G/\Gamma = UF\Gamma$ is compact. ☐

When G is a locally compact second countable group, we prove a very useful criterion to ensure that a discrete subgroup $\Gamma < G$ is a lattice.

Theorem 1.17. *Let G be any locally compact second countable group and $\Gamma < G$ any discrete subgroup. The following assertions are equivalent:*

(i) *$\Gamma < G$ is a lattice.*
(ii) *G is unimodular and there is a Borel fundamental domain $\mathcal{F} \subset G$ for the right translation action $\Gamma \curvearrowright G$ such that $0 < m_G(\mathcal{F}) < +\infty$.*
(iii) *G is unimodular and there is a Borel subset $\mathfrak{S} \subset G$ such that $\mathfrak{S} \cdot \Gamma = G$ and $0 < m_G(\mathfrak{S}) < +\infty$.*

Proof. Recall that since G is a locally compact second countable group, the discrete subgroup $\Gamma < G$ is necessarily countable.

(i) ⇒ (ii) We already know that G is unimodular by Proposition 1.15. Denote by $\nu \in \operatorname{Prob}(G/\Gamma)$ the unique G-invariant regular Borel probability measure. Denote by m_G the unique Haar measure on G satisfying (1.3). Since G is locally compact second countable, (1.3) holds for every nonnegative Borel function $f : G \to \mathbb{R}_+$. In particular, for $f = \mathbf{1}_{\mathcal{F}}$, we have $\overline{f} \equiv 1$ and so

$$m_G(\mathcal{F}) = \int_G \mathbf{1}_{\mathcal{F}}(h)\, \mathrm{d}m_G(h) = \int_{G/\Gamma} 1 \, \mathrm{d}\nu(h\Gamma) = 1 < +\infty.$$

Since $m_G(G) > 0$, $G = \bigcup_{\gamma \in \Gamma} \mathcal{F}\gamma$ and $m_G(\mathcal{F}\gamma) = m_G(\mathcal{F})$ for every $\gamma \in \Gamma$, we also have $m_G(\mathcal{F}) > 0$.

(ii) ⇒ (iii) It is trivial.

(iii) ⇒ (i) Following the proof of Proposition 1.14 and since m_G is right invariant, we may consider the well-defined left invariant linear functional

$$\Phi : \mathrm{C}_c(G/\Gamma) \to \mathbb{C} : \overline{f} \mapsto \int_G f(g)\, \mathrm{d}m_G(g).$$

By Riesz's representation theorem, there exists a unique G-invariant regular Borel measure ν on G/Γ such that (1.3) holds. Then we have

$$\nu(G/\Gamma) = \sup \left\{ \int_{G/\Gamma} \varphi(h\Gamma)\,\mathrm{d}\nu(h\Gamma) \mid \varphi \in \mathrm{C}_c(G/\Gamma),\ 0 \le f \le \mathbf{1}_{G/\Gamma} \right\}.$$

Let $\varphi \in \mathrm{C}_c(G/\Gamma)$ be such that $0 \le \varphi \le \mathbf{1}_{G/\Gamma}$ and choose $f \in \mathrm{C}_c(G)_+$ such that $\varphi = \overline{f}$. Since $G = \bigcup_{\gamma\in\Gamma} \mathfrak{S}\gamma$, using Fubini's theorem, we have

$$\begin{aligned}
\int_{G/\Gamma} \varphi(h\Gamma)\,\mathrm{d}\nu(h\Gamma) &= \int_G f(h)\,\mathrm{d}m_G(h) \\
&\le \sum_{\gamma\in\Gamma} \int_{\mathfrak{S}\gamma} f(h)\,\mathrm{d}m_G(h) \\
&= \sum_{\gamma\in\Gamma} \int_{\mathfrak{S}} f(h\gamma)\,\mathrm{d}m_G(h) \\
&= \int_{\mathfrak{S}} \sum_{\gamma\in\Gamma} f(h\gamma)\,\mathrm{d}m_G(h) \\
&\le \int_{\mathfrak{S}} 1\,\mathrm{d}m_G(h) \\
&= m_G(\mathfrak{S}) < +\infty.
\end{aligned}$$

This implies that $\nu(G/\Gamma) \le m_G(\mathfrak{S}) < +\infty$. □

Let us point out that when $\Gamma < G$ is a lattice then all Borel fundamental domains for the right translation action $\Gamma \curvearrowright G$ have the same finite Haar measure. Indeed, whenever $\mathcal{F}_1, \mathcal{F}_2 \subset G$ are Borel fundamental domains, since the Haar measure m_G on G is right invariant, we have

$$\begin{aligned}
m_G(\mathcal{F}_1) &= \sum_{\gamma\in\Gamma} m_G(\mathcal{F}_1 \cap \mathcal{F}_2\gamma) \\
&= \sum_{\gamma\in\Gamma} m_G(\mathcal{F}_1\gamma^{-1} \cap \mathcal{F}_2) \\
&= m_G(\mathcal{F}_2).
\end{aligned}$$

Examples 1.18. Here are some examples of lattices in locally compact groups.

(i) For every $d \ge 1$, the discrete subgroup $\mathbb{Z}^d < \mathbb{R}^d$ is a uniform lattice.
(ii) More generally, any lattice $\Gamma < G$ in a locally compact second countable abelian group G is necessarily uniform.
(iii) The discrete Heisenberg group $\mathrm{H}_3(\mathbb{Z}) < \mathrm{H}_3(\mathbb{R})$ is a uniform lattice in the continuous Heisenberg group $\mathrm{H}_3(\mathbb{R})$:

$$\mathrm{H}_3(\mathbb{Z}) \doteq \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{Z} \right\}$$

$$\mathrm{H}_3(\mathbb{R}) \doteq \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\}.$$

(iv) More generally, any lattice $\Gamma < G$ in a locally compact second countable nilpotent group G is necessarily uniform.

3. $\mathrm{SL}_d(\mathbb{Z})$ is a lattice in $\mathrm{SL}_d(\mathbb{R})$, $d \geq 2$

In this section, we prove the following theorem due to Minkowski.

THEOREM 1.19 (Minkowski). *For every $d \geq 2$, the discrete subgroup $\mathrm{SL}_d(\mathbb{Z}) < \mathrm{SL}_d(\mathbb{R})$ is a nonuniform lattice.*

Before proving Theorem 1.19, we need to prove some preliminary results that are also of independent interest.

Let $d \geq 1$. Endow $\mathbb{R}^d$ with its canonical euclidean structure. Denote by $K \doteq \mathrm{SO}_d(\mathbb{R}) < \mathrm{SL}_d(\mathbb{R})$ the special orthogonal subgroup and observe that $K < \mathrm{SL}_d(\mathbb{R})$ is compact. Denote by $A < \mathrm{SL}_d(\mathbb{R})$ the subgroup of diagonal matrices with positive entries, that is,

$$A \doteq \{a = \mathrm{diag}(\lambda_1, \dots, \lambda_d) \mid \lambda_1, \dots, \lambda_d > 0,\ \lambda_1 \cdots \lambda_d = 1\} < \mathrm{SL}_d(\mathbb{R}).$$

Denote by $N \doteq \mathrm{T}_d(\mathbb{R}) < \mathrm{SL}_d(\mathbb{R})$ the strict upper triangular subgroup as in Example 1.10(vi).

LEMMA 1.20 (Iwasawa decomposition). *The map $K \times A \times N \to \mathrm{SL}_d(\mathbb{R}) : (k, a, n) \mapsto kan$ is a homeomorphism. We simply write $\mathrm{SL}_d(\mathbb{R}) = K \cdot A \cdot N$.*

PROOF. Denote by $(e_1, \dots, e_d)$ the canonical basis of $\mathbb{R}^d$. The map $\Psi : K \times A \times N \to \mathrm{SL}_d(\mathbb{R}) : (k, a, n) \mapsto kan$ is clearly continuous. Conversely, let $g \in \mathrm{SL}_d(\mathbb{R})$ be any element and write $v_i = ge_i \in \mathbb{R}^d$ for every $1 \leq i \leq d$. By Gram–Schmidt's orthogonalization process, set $w_1 = v_1$ and $w_{i+1} = v_{i+1} - P_{V_i}(v_{i+1})$ where $V_i = \mathrm{Vect}(v_1, \dots, v_i)$ for every $1 \leq i \leq d-1$. Then $(\frac{w_1}{\|w_1\|}, \dots, \frac{w_d}{\|w_d\|})$ is an orthonormal basis for $\mathbb{R}^d$ and we may find $k \in \mathrm{O}_d(\mathbb{R})$ such that $ke_i = \frac{w_i}{\|w_i\|}$ for every $1 \leq i \leq d$. Then the matrix $k^{-1}g$ is upper triangular and $(k^{-1}g)_{ii} = \|w_i\|$ for every $1 \leq i \leq d$. It follows that $\det(k^{-1}) = \det(k^{-1}g) = \|w_1\| \cdots \|w_d\| > 0$ and hence $k \in \mathrm{SO}_d(\mathbb{R})$. Letting $a = \mathrm{diag}(\|w_1\|, \dots, \|w_d\|) \in A$, we have $g = kan$ and the map $\mathrm{SL}_d(\mathbb{R}) \to K \times A \times N : g \mapsto (k, a, n)$ is continuous. Since its inverse is Ψ, we have showed that $\Psi : K \times A \times N \to \mathrm{SL}_d(\mathbb{R}) : (k, a, n) \mapsto kan$ is a homeomorphism. $\square$

LEMMA 1.21. *Endow $(K, \mathrm{d}k)$, $(A, \mathrm{d}a)$, $(N, \mathrm{d}n)$ with their respective Haar measure. Then the pushforward measure of*

$$\prod_{1 \leq i < j \leq d} \frac{\lambda_i}{\lambda_j}\, \mathrm{d}k\, \mathrm{d}a\, \mathrm{d}n$$

under the map $K \times A \times N \to \mathrm{SL}_d(\mathbb{R}) : (k, a, n) \mapsto kan$ is a Haar measure on $\mathrm{SL}_d(\mathbb{R})$.

Proof. Consider the product map $\Psi : K \times AN \to \mathrm{SL}_d(\mathbb{R}) : (k,p) \mapsto k^{-1}p$. Since $\mathrm{SL}_d(\mathbb{R})$ is unimodular, the regular Borel measure $(\Psi^{-1})_* m_{\mathrm{SL}_d(\mathbb{R})}$ on $K \times AN$ is right invariant. Then $(\Psi^{-1})_* m_{\mathrm{SL}_d(\mathbb{R})}$ is a right invariant Haar measure on the locally compact second countable group $K \times AN$ and hence $(\Psi^{-1})_* m_{\mathrm{SL}_d(\mathbb{R})} = \mu_K \otimes \mu_{AN}$ where μ_K is a right invariant Haar measure on K and μ_{AN} is a right invariant Haar measure on AN. Since K is compact, μ_K is also left invariant and hence we may assume that $\mathrm{d}\mu_K(k) = \mathrm{d}k$. It remains to prove that $\prod_{1 \leq i < j \leq d} \frac{\lambda_i}{\lambda_j} \, \mathrm{d}a \, \mathrm{d}n$ is a right invariant Haar measure on AN.

As explained in Examples 1.10(vi), we may assume that $\mathrm{d}m_N(n) \doteq \mathrm{d}n = \prod_{1 \leq i < j \leq d} \mathrm{d}n_{ij}$. Observe that $N \lhd AN$ is a normal subgroup and define the conjugation action $\mathrm{Ad} : A \curvearrowright N$ by $\mathrm{Ad}(a)(n) = ana^{-1}$ for $a \in A$, $n \in N$. Then $AN = A \ltimes N$ and $\mathrm{d}a \, \mathrm{d}n$ is a left invariant measure on AN by Proposition 1.9. A simple calculation shows that $\mathrm{Ad}(a)_* m_N = (\prod_{1 \leq i < j \leq d} \frac{\lambda_i}{\lambda_j})^{-1} \cdot m_N$. Then Proposition 1.9 implies that $\prod_{1 \leq i < j \leq d} \frac{\lambda_i}{\lambda_j} \, \mathrm{d}a \, \mathrm{d}n$ is a right invariant Haar measure on AN. $\square$

For all $t, u > 0$, set

$$\begin{aligned} A_t &\doteq \{a = \mathrm{diag}(\lambda_1, \dots, \lambda_d) \in A \mid \forall 1 \leq i \leq d-1, \lambda_i \leq t\lambda_{i+1}\} \\ N_u &\doteq \{n = (n_{ij})_{ij} \in N \mid \forall 1 \leq i < j \leq d, |n_{ij}| \leq u\} \\ \mathfrak{S}_{t,u} &\doteq K \cdot A_t \cdot N_u. \end{aligned}$$

The Borel subset $\mathfrak{S}_{t,u} \subset G$ is called a *Siegel domain*. We now have all the tools to prove Theorem 1.19.

Proof of Theorem 1.19. For every $t \geq \frac{2}{\sqrt{3}}$ and every $u \geq \frac{1}{2}$, we show that $\mathrm{SL}_d(\mathbb{R}) = \mathfrak{S}_{t,u} \cdot \mathrm{SL}_d(\mathbb{Z})$ and that $\mathfrak{S}_{t,u}$ has finite Haar measure. By Theorem 1.17, this implies that $\mathrm{SL}_d(\mathbb{Z}) < \mathrm{SL}_d(\mathbb{R})$ is a lattice. We divide the proof into a series of claims.

Claim 1.22. For all $t, u > 0$, the Siegel domain $\mathfrak{S}_{t,u}$ has finite Haar measure.

Indeed, note that since K and N_u are both compact in $\mathrm{SL}_d(\mathbb{R})$, using Lemma 1.21 it suffices to prove that

$$\kappa_t \doteq \int_{A_t} \prod_{1 \leq i < j \leq d} \frac{\lambda_i}{\lambda_j} \, \mathrm{d}a < +\infty.$$

Observe that the map

$$\Theta : A \to \mathbb{R}^{d-1} : \mathrm{diag}(\lambda_1, \dots, \lambda_d) \mapsto \left(\log \frac{\lambda_2}{\lambda_1}, \dots, \log \frac{\lambda_d}{\lambda_{d-1}} \right)$$

is a topological group isomorphism. We may choose the Haar measure $\mathrm{d}a$ on A that is the pushforward of the Lebesgue measure on $\mathbb{R}^{d-1}$ by Θ^{-1}. We then have

$$\begin{aligned}\kappa_t &= \int_{\mathbb{R}^{d-1}} \prod_{1\le i<j\le d} \exp(-(s_i+\cdots+s_{j-1}))\mathbf{1}_{\{s_1,\ldots,s_{d-1}\ge -\log t\}}\, \mathrm{d}s_1\cdots \mathrm{d}s_{d-1} \\ &= \prod_{k=1}^{d-1}\int_{-\log t}^{+\infty} \exp(-k(d-k)s_k)\,\mathrm{d}s_k < +\infty.\end{aligned}$$

Claim 1.23. For every $u \ge \frac{1}{2}$, we have $N = N_u \cdot (N \cap \mathrm{SL}_d(\mathbb{Z}))$.

Indeed, it suffices to prove Claim 1.23 for $u = \frac{1}{2}$. We proceed by induction over $d \ge 1$. For $d = 1$, there is nothing to prove. Assume that the result is true for $d-1 \ge 1$ and let us prove it for d. Let $n \in N = \mathrm{T}_d(\mathbb{R})$ be any element that we write

$$n = \begin{pmatrix} 1 & * \\ 0 & n_0 \end{pmatrix} \quad \text{where} \quad n_0 \in \mathrm{T}_{d-1}(\mathbb{R}).$$

By induction hypothesis, there exists $\gamma_0 \in \mathrm{T}_{d-1}(\mathbb{R}) \cap \mathrm{SL}_{d-1}(\mathbb{Z})$ such that $n_1 \doteq n_0\gamma_0^{-1} \in \mathrm{T}_{d-1}(\mathbb{R})_{1/2}$. Write

$$n\begin{pmatrix} 1 & 0 \\ 0 & \gamma_0^{-1} \end{pmatrix} = \begin{pmatrix} 1 & x \\ 0 & n_1 \end{pmatrix} \quad \text{where} \quad x \in \mathbb{R}^{d-1}.$$

Choose $y \in \mathbb{Z}^{d-1}$ such that $x - y \in [-1/2, 1/2]^{d-1}$. Then

$$\begin{aligned} n &= \begin{pmatrix} 1 & x \\ 0 & n_1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \gamma_0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & x-y \\ 0 & n_1 \end{pmatrix}\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \gamma_0 \end{pmatrix}\end{aligned}$$

where

$$\begin{pmatrix} 1 & x-y \\ 0 & n_1 \end{pmatrix} \in N_{1/2} \quad \text{and} \quad \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \gamma_0 \end{pmatrix} \in N \cap \mathrm{SL}_d(\mathbb{Z}).$$

This shows the result is true for d and finishes the proof of Claim 1.23.

Claim 1.24. For every $t \ge \frac{2}{\sqrt{3}}$, we have $\mathrm{SL}_d(\mathbb{R}) = K \cdot A_t \cdot N \cdot \mathrm{SL}_d(\mathbb{Z})$.

Indeed, it suffices to prove Claim 1.24 for $t = \frac{2}{\sqrt{3}}$. We proceed by induction over $d \ge 1$. For $d = 1$, there is nothing to prove. Assume that the result is true for $d-1 \ge 1$ and let us prove it for d. Denote by $(e_1, \ldots, e_d)$ the canonical basis of $\mathbb{R}^d$. Let $g \in \mathrm{SL}_d(\mathbb{R})$ be any element. Since $\Lambda = g\mathbb{Z}^d$ is a lattice in $\mathbb{R}^d$, there must exist a vector $v_1 \in \Lambda \setminus \{0\}$ such that

$$\|v_1\| = \min\{\|v\| \mid v \in \Lambda \setminus \{0\}\}.$$

By minimality of the norm of $v_1 \in \Lambda\setminus\{0\}$, we may find $v_2, \ldots, v_d \in \Lambda \setminus \{0\}$ such that $(v_1, \ldots, v_d)$ is a basis of Λ (see e.g. [Ca71, Corollary I.3]). Up to further replacing v_1 by $-v_1$, there exists $\gamma \in \mathrm{SL}_d(\mathbb{Z})$ such that $\gamma e_i = g^{-1}v_i$ for every $1 \le i \le d$. Note that $g\gamma e_1 = v_1$.

Next, consider the Iwasawa decomposition $g\gamma = kan$ and write

$$an = \begin{pmatrix} \lambda & * \\ 0 & \lambda^{-1} g_0 \end{pmatrix} \quad \text{where} \quad \lambda \in \mathbb{R}^*,\ g_0 \in \mathrm{SL}_{d-1}(\mathbb{R}).$$

By induction hypothesis, there exist $k_0 \in \mathrm{SO}_{d-1}(\mathbb{R})$ and $\gamma_0 \in \mathrm{SL}_{d-1}(\mathbb{Z})$ such that $k_0^{-1} g_0 \gamma_0^{-1} \in (A_{d-1})_{2/\sqrt{3}} \cdot \mathrm{T}_{d-1}(\mathbb{R})$. If we consider

$$h = \begin{pmatrix} 1 & 0 \\ 0 & k_0^{-1} \end{pmatrix} k^{-1} g\gamma \begin{pmatrix} 1 & 0 \\ 0 & \gamma_0^{-1} \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1}\, k_0^{-1} g_0 \gamma_0^{-1} \end{pmatrix} \in AN$$

we obtain that the diagonal coefficients of h satisfy $h_{i,i} \leq \frac{2}{\sqrt{3}} h_{i+1,i+1}$ for every $2 \leq i \leq d-1$. It remains to prove that $h_{1,1} \leq \frac{2}{\sqrt{3}} h_{2,2}$. Observe that for every $w \in \mathbb{Z}^d \setminus \{0\}$, we have

$$\|he_1\| = \|g\gamma \begin{pmatrix} 1 & 0 \\ 0 & \gamma_0^{-1} \end{pmatrix} e_1\| = \|g\gamma e_1\| = \|v_1\| \leq \|g\gamma \begin{pmatrix} 1 & 0 \\ 0 & \gamma_0^{-1} \end{pmatrix} w\| = \|hw\|.$$

Using Claim 1.23, write $h = \mathrm{diag}(h_{11}, \ldots, h_{dd}) n_1 \gamma_1$ where $n_1 \in N_{1/2}$ and $\gamma_1 \in N \cap \mathrm{SL}_d(\mathbb{Z})$. Then $he_1 = \mathrm{diag}(h_{11}, \ldots, h_{dd}) e_1 = h_{11} e_1$ and with $w = \gamma_1^{-1} e_2 \in \mathbb{Z}^d \setminus \{0\}$, we have $hw = \mathrm{diag}(h_{11}, \ldots, h_{dd}) n_1 e_2 = h_{11} n_{12} e_1 + h_{22} e_2$. Then we obtain

$$h_{11}^2 = \|he_1\|^2 \leq \|hw\|^2 = h_{11}^2 n_{12}^2 + h_{22}^2 \leq \frac{1}{4} h_{11}^2 + h_{22}^2$$

and so $h_{11}^2 \leq \frac{4}{3} h_{22}^2$. This finishes the proof of Claim 1.24.

A combination of Claims 1.22, 1.23, 1.24 and Theorem 1.17 implies that $\mathrm{SL}_d(\mathbb{Z}) < \mathrm{SL}_d(\mathbb{R})$ is a lattice.

It remains to prove that $\mathrm{SL}_d(\mathbb{Z}) < \mathrm{SL}_d(\mathbb{R})$ is nonuniform. Indeed, regard $\mathrm{SL}_2(\mathbb{R}) < \mathrm{SL}_d(\mathbb{R})$ as a subgroup in the top left corner and set

$$\gamma \doteq \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) < \mathrm{SL}_d(\mathbb{Z}).$$

Then a simple calculation shows that

$$g_n \gamma^n g_n^{-1} = \begin{pmatrix} 1 & n^{-1} \\ 0 & 1 \end{pmatrix} \to e \quad \text{with} \quad g_n \doteq \begin{pmatrix} n^{-1} & 0 \\ 0 & n \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}) < \mathrm{SL}_d(\mathbb{R}).$$

Then Proposition 1.16 implies that $\mathrm{SL}_d(\mathbb{Z}) < \mathrm{SL}_d(\mathbb{R})$ is nonuniform. $\square$

4. More examples of lattices

In this expository section, we provide more examples of lattices in locally compact groups. The main examples of lattices we discuss arise from arithmetic subgroups of algebraic groups.

We first introduce some terminology. We say that a connected linear algebraic group $\mathbf{G} < \mathrm{GL}_n(\mathbb{C})$ is *defined over* $\mathbb{Q}$ or is a $\mathbb{Q}$-*group* if the ideal $\mathcal{J}(\mathbf{G}) \subset \mathbb{C}[g_{11}, \ldots, g_{ij}, \ldots, g_{nn}, T]$ of all polynomials vanishing on $\mathbf{G}$ is spanned by $\mathcal{J}_{\mathbb{Q}}(\mathbf{G}) \doteq \mathcal{J}(\mathbf{G}) \cap \mathbb{Q}[g_{11}, \ldots, g_{ij}, \ldots, g_{nn}, T]$ over $\mathbb{C}$. Moreover, we say that $\mathbf{G}$ is

- *semisimple* if its maximal connected algebraic solvable normal subgroup is trivial.
- *almost simple* if the only proper algebraic normal subgroups are finite.

We then say that $\mathbf{G}(\mathbb{Z}) \doteq \mathbf{G} \cap \mathrm{GL}_d(\mathbb{Z})$ is an *arithmetic* group.

EXAMPLE 1.25. For every $d \geq 2$, the special linear group SL_d is a connected almost simple semisimple algebraic $\mathbb{Q}$-group.

The next theorem is a particular case of a general result due to Borel–Harish-Chandra showing that arithmetic groups are lattices.

THEOREM 1.26 (Borel–Harish-Chandra [**BHC61**]). *Let $\mathbf{G}$ be any connected semisimple algebraic $\mathbb{Q}$-group. Then $\mathbf{G}(\mathbb{Z}) < \mathbf{G}(\mathbb{R})$ is a nonuniform lattice.*

One can then view Theorem 1.26 as a generalization of Theorem 1.19.

In these lecture notes, we will be interested in discrete groups that arise as lattices in product groups. In that respect, we introduce the following terminology. Let $r \geq 2$ and $G_1, \dots, G_r$ be any locally compact groups. Set $G = G_1 \times \cdots \times G_r$. For every $1 \leq i \leq r$, denote by $p_i : G \to G_i$ the canonical factor map.

DEFINITION 1.27. Let $\Gamma < G$ be any discrete subgroup. We say that $\Gamma < G$ is *irreducible* if for every $1 \leq i \leq r$, the image $p_i(\Gamma)$ is dense in G_i.

Let us point out that Definition 1.27 is not really restrictive. Indeed, whenever $\Gamma < G$ is a discrete subgroup, letting $H_i = \overline{p_i(\Gamma)}$ for every $1 \leq i \leq r$, we may regard Γ as a discrete and irreducible subgroup of the locally compact group $H = H_1 \times \cdots \times H_r$.

EXAMPLE 1.28. Here are some examples of discrete irreducible subgroups $\Gamma < G$ in locally compact groups.

(i) Let $q \geq 2$ be any square-free integer. Define the field automorphism $\sigma : \mathbb{Q}(\sqrt{q}) \to \mathbb{Q}(\sqrt{q}) : x + y\sqrt{q} \mapsto x - y\sqrt{q}$. For every $d \geq 2$, the subgroup
$$\Gamma \doteq \{(g, g^\sigma) \mid g \in \mathrm{SL}_d(\mathbb{Z}[\sqrt{q}])\} < \mathrm{SL}_d(\mathbb{R}) \times \mathrm{SL}_d(\mathbb{R})$$
is discrete and irreducible. Write $\mathrm{SL}_d(\mathbb{Z}[\sqrt{q}]) < \mathrm{SL}_d(\mathbb{R}) \times \mathrm{SL}_d(\mathbb{R})$.

(ii) Let $p \in \mathcal{P}$ be any prime. For every $d \geq 2$, the subgroup
$$\Gamma \doteq \left\{(g, g) \mid g \in \mathrm{SL}_d(\mathbb{Z}[p^{-1}])\right\} < \mathrm{SL}_d(\mathbb{R}) \times \mathrm{SL}_d(\mathbb{Q}_p)$$
is discrete and irreducible. Write $\mathrm{SL}_d(\mathbb{Z}[p^{-1}]) < \mathrm{SL}_d(\mathbb{R}) \times \mathrm{SL}_d(\mathbb{Q}_p)$.

Borel–Harish-Chandra's results [**BHC61**] provide many examples of lattices in algebraic groups. We refer the reader to [**Ma91**, Chapter IX] and [**Be09**, §2] for further details.

EXAMPLES 1.29. Let $d \geq 2$.

(i) The discrete subgroup $\mathrm{SL}_d(\mathbb{Z}) < \mathrm{SL}_d(\mathbb{R})$ is a nonuniform lattice (see Theorem 1.19).

(ii) For every square-free integer $q \geq 2$, the discrete subgroup

$$\mathrm{SL}_d(\mathbb{Z}[\sqrt{q}]) < \mathrm{SL}_d(\mathbb{R}) \times \mathrm{SL}_d(\mathbb{R})$$

is a nonuniform irreducible lattice.

(iii) For every prime $p \in \mathcal{P}$, the discrete subgroup

$$\mathrm{SL}_d(\mathbb{Z}[p^{-1}]) < \mathrm{SL}_d(\mathbb{R}) \times \mathrm{SL}_d(\mathbb{Q}_p)$$

is a nonuniform irreducible lattice.

(iv) More generally, for every finite set of primes $S = \{p_1, \dots, p_r\} \subset \mathcal{P}$, the discrete subgroup

$$\mathrm{SL}_d(\mathbb{Z}[S^{-1}]) < \mathrm{SL}_d(\mathbb{R}) \times \mathrm{SL}_d(\mathbb{Q}_{p_1}) \times \cdots \times \mathrm{SL}_d(\mathbb{Q}_{p_r})$$

is a nonuniform irreducible lattice.

(v) For every prime $p \in \mathcal{P}$, denote by $\mathbf{F}_p((t))$ the field of formal power series in one variable t over the finite field $\mathbf{F}_p$ and by $\mathbf{F}_p[t^{-1}] \subset \mathbf{F}_p((t))$ the polynomial ring in one variable t^{-1}. Then the discrete subgroup $\mathrm{SL}_d(\mathbf{F}_p[t^{-1}]) < \mathrm{SL}_d(\mathbf{F}_p((t)))$ is a nonuniform lattice.

(vi) Let $d \geq 3$ and $p \geq q \geq 1$ such that $p + q = d$. Define

$$\begin{aligned} J_{p,q} &\doteq \begin{pmatrix} 1_p & 0 \\ 0 & -\sqrt{2}\,1_q \end{pmatrix} \\ \Gamma &\doteq \left\{ g \in \mathrm{SL}_d(\mathbb{Z}[\sqrt{2}]) \mid g J_{p,q} {}^t g = J_{p,q} \right\} \\ G &\doteq \left\{ g \in \mathrm{SL}_d(\mathbb{R}) \mid g J_{p,q} {}^t g = J_{p,q} \right\}. \end{aligned}$$

Then $\Gamma < G$ is a uniform lattice.

CHAPTER 2

Group unitary representation theory

In this chapter, we present an introduction to unitary representation theory for locally compact groups. We define and study the notions of amenability and Kazhdan's property (T). We prove that $\mathrm{SL}_d(\mathbb{R})$ has the Howe–Moore property for every $d \geq 2$. We also prove that $\mathrm{SL}_d(\mathbb{R})$ and its lattice $\mathrm{SL}_d(\mathbb{Z})$ have Kazhdan's property (T) for every $d \geq 3$.

1. Generalities on unitary representations

Let $(\mathcal{H}, \langle \cdot , \cdot \rangle)$ be any (complex) Hilbert space. We always assume that $\langle \cdot , \cdot \rangle$ is conjugate linear in the second variable. We denote by

$$\mathcal{U}(\mathcal{H}) = \{u \in \mathbb{B}(\mathcal{H}) \mid u^* u = uu^* = 1_{\mathcal{H}}\}$$

the group of unitary operators on $\mathcal{H}$. We simply write $1 = 1_{\mathcal{H}}$. We endow $\mathcal{U}(\mathcal{H})$ with the *strong operator topology* defined as the initial topology on $\mathcal{U}(\mathcal{H})$ that makes the maps $\mathcal{U}(\mathcal{H}) \to \mathbb{R} : u \mapsto \|(u-1)\xi\|$ continuous for all $\xi \in \mathcal{H}$. Then $\mathcal{U}(\mathcal{H})$ is a topological group but $\mathcal{U}(\mathcal{H})$ need not be locally compact. When $\mathcal{H}$ is separable, $\mathcal{U}(\mathcal{H})$ is a Polish group.

Definition 2.1. Let G be any locally compact group. We say that the mapping $\pi : G \to \mathcal{U}(\mathcal{H}_\pi)$ is a *strongly continuous unitary representation* if the following conditions hold:

(i) $\pi : G \to \mathcal{U}(\mathcal{H}_\pi)$ is a group homomorphism.
(ii) $\pi : G \to \mathcal{U}(\mathcal{H}_\pi)$ is strongly continuous, meaning that π is a continuous map when $\mathcal{U}(\mathcal{H}_\pi)$ is endowed with the strong operator topology as above.

When $\pi : G \to \mathcal{U}(\mathcal{H}_\pi)$ only satisfies condition (i), we simply say that π is a *unitary representation*. When G is discrete, condition (ii) is trivially satisfied.

The next result shows that in order to prove that the unitary representation $\pi : G \to \mathcal{U}(\mathcal{H}_\pi)$ is strongly continuous, it is enough to show that the coefficients of π are measurable functions.

Lemma 2.2. *Let G be any locally compact group, $\mathcal{H}_\pi$ any separable Hilbert space and $\pi : G \to \mathcal{U}(\mathcal{H}_\pi)$ any unitary representation. Assume that for all $\xi, \eta \in \mathcal{H}_\pi$, the map $\varphi_{\xi,\eta} : G \to \mathbb{C} : g \mapsto \langle \pi(g)\xi, \eta\rangle$ is measurable. Then π is strongly continuous.*

PROOF. Let $\xi \in \mathcal{H}_\pi$ be any vector. It suffices to show that the map $G \to \mathcal{H}_\pi : g \mapsto \pi(g)\xi$ is continuous at $e \in G$. Let $Q \subset G$ be any symmetric compact neighborhood of $e \in G$. Consider the compactly generated open subgroup $H \doteq \bigcup_{n \geq 1} Q^n < G$. It further suffices to show that the map $H \to \mathcal{H}_\pi : g \mapsto \pi(g)\xi$ is continuous at $e \in H$. Thus, we may as well assume that G is σ-compact.

As usual, we denote by m_G a left invariant Haar measure on G. Let $\varepsilon > 0$ and set $B \doteq \{g \in G \mid \|\pi(g)\xi - \xi\| < \varepsilon/2\}$. Then $B \subset G$ is a measurable subset since $B = \{g \in G \mid 2\Re(\langle \pi(g)\xi, \xi\rangle) > 2\|\xi\|^2 - \varepsilon^2/4\}$. Moreover, we have $B^{-1} = B$ and $B^2 = BB^{-1} \subset \{g \in G \mid \|\pi(g)\xi - \xi\| < \varepsilon\}$. Since $\pi(G)\xi \subset \mathcal{H}_\pi$ is separable, there exists a sequence $(g_n)_{n\in\mathbb{N}}$ in G such that $(\pi(g_n)\xi)_{n\in\mathbb{N}}$ is dense in $\pi(G)\xi$. This implies that $\bigcup_{n\in\mathbb{N}} g_n B = G$ and so $m_G(B) > 0$. Since G is σ-compact, up to replacing B by $B \cap K$ for a suitable symmetric compact subset, we may further assume that $B = B^{-1}$, $B \subset K$ and $0 < m_G(B) < +\infty$. Then $\mathbf{1}_B \in \mathrm{L}^2(G, \mathcal{B}(G), m_G)$ and $\varphi = \mathbf{1}_B * \mathbf{1}_B \in \mathrm{C}_c(G)$ with $\operatorname{supp}(\varphi) \subset B^2 \subset K^2$. Since $\varphi(e) = m_G(B) > 0$, the subset $U = \varphi^{-1}(0, +\infty)$ is open, $e \in U$ and $U \subset BB \subset \{g \in G \mid \|\pi(g)\xi - \xi\| < \varepsilon\}$. $\square$

DEFINITION 2.3. Let G be any locally compact group and $\pi : G \to \mathcal{U}(\mathcal{H}_\pi)$ any strongly continuous unitary representation. We say that

- π has *invariant vectors* and we write $1_G \subset \pi$ if the subspace of $\pi(G)$-invariant vectors
$$(\mathcal{H}_\pi)^G \doteq \{\xi \in \mathcal{H}_\pi \mid \forall g \in G, \pi(g)\xi = \xi\}$$
is nonzero. Otherwise, we say that π is *ergodic* and we write $1_G \not\subset \pi$.
- π has *almost invariant vectors* and we write $1_G \prec \pi$ if for every $\varepsilon > 0$ and every compact subset $Q \subset G$, there exists a unit vector $\xi \in \mathcal{H}_\pi$ such that
$$\sup_{g\in Q} \|\pi(g)\xi - \xi\| < \varepsilon.$$
Otherwise, we say that π has *spectral gap* and we write $1_G \not\prec \pi$.

It is clear that if $1_G \subset \pi$, then $1_G \prec \pi$.

For every $i \in \{1, 2\}$, let $\pi_i : G \to \mathcal{U}(\mathcal{H}_{\pi_i})$ be any strongly continuous unitary representation. We say that π_1 and π_2 are *unitarily equivalent* if there exists a unitary operator $U : \mathcal{H}_{\pi_1} \to \mathcal{H}_{\pi_2}$ such that for every $g \in G$, we have $\pi_2(g) = U\pi_1(g)U^*$. In this situation, we will identify π_1 with π_2.

1.1. Examples of unitary representations. Let G be any locally compact group.

The left regular representation λ_G. Let m_G be any left invariant Haar measure on G and simply denote by $\mathrm{L}^2(G) = \mathrm{L}^2(G, \mathcal{B}(G), m_G)$ the corresponding Hilbert space of L^2-integrable functions on G. Define the *left regular representation* $\lambda_G : G \to \mathcal{U}(\mathrm{L}^2(G))$ by the formula
$$\forall g \in G, \forall \xi \in \mathrm{L}^2(G), \quad (\lambda_G(g)\xi)(h) = \xi(g^{-1}h).$$

The left regular representation $\lambda_G : G \to \mathcal{U}(\mathrm{L}^2(G))$ is a strongly continuous unitary representation. This follows from the well known facts that the subspace $\mathrm{C}_c(G)$ of compactly supported continuous functions on G is $\|\cdot\|_2$-dense in $\mathrm{L}^2(G)$ and the left translation action $\lambda : G \curvearrowright \mathrm{C}_c(G)$ is $\|\cdot\|_\infty$-continuous (see Lemma 1.8).

Proposition 2.4. *Keep the same notation as above. Then $1_G \subset \lambda_G$ if and only if G is compact.*

Proof. If G is compact, then the left invariant Haar measure m_G is finite. This implies that the constant function $\mathbf{1}_G$ belongs to $\mathrm{L}^2(G)$ and is $\lambda_G(G)$-invariant. Conversely, assume that there exists a nonzero $\lambda_G(G)$-invariant vector $\xi \in \mathrm{L}^2(G)$.

Claim 2.5. There exists a σ-compact open subgroup $H < G$ such that $\xi = \mathbf{1}_H \xi$.

Indeed, define the measurable subsets $B \doteq \{h \in G \mid \xi(h) \neq 0\}$ and $B_n \doteq \{h \in G \mid |\xi(h)| \geq n^{-1}\}$ for every $n \geq 1$. Then $B = \bigcup_{n\geq 1} B_n$ and $m_G(B_n) < +\infty$ for every $n \geq 1$. By regularity, for every $n \geq 1$, there exists an open set $U_n \subset G$ such that $B_n \subset U_n$ and $m_G(U_n) < +\infty$. To prove the claim, it suffices to show that every open set $U \subset G$ with finite Haar measure is contained in a σ-compact open subgroup $H < G$.

Let $U \subset G$ be any nonempty open set such that $m_G(U) < +\infty$. Let $L < G$ be any σ-compact open subgroup. Then the set $\Lambda \doteq \{g \in G \setminus L \mid U \cap gL \neq \emptyset\}$ is at most countable. Letting $H < G$ be the subgroup generated by L and Λ, we have that $U \subset H$ and $H < G$ is σ-compact and open. This finishes the proof of Claim 2.5.

Using Claim 2.5 and the assumption, for every $g \in G$, we have

$$\mathbf{1}_H \xi = \xi = \lambda_G(g)\xi = \lambda_G(g)(\mathbf{1}_H \xi) = \mathbf{1}_{gH} \xi.$$

Since $\xi \neq 0$, we have $\mathbf{1}_{gH} = \mathbf{1}_H$ and hence $m_G(gH \triangle H) = 0$ for every $g \in G$. Since $H < G$ is open, it follows that $gH = H$ for every $g \in G$ and hence $H = G$. This shows that G is σ-compact.

We may now apply Fubini's theorem. Indeed, since for every $g \in G$ and m_G-almost every $h \in G$, we have $\xi(g^{-1}h) = \xi(h)$, Fubini's theorem implies that there exists $h \in G$ such that for m_G-almost every $g \in G$, we have $\xi(g^{-1}h) = \xi(h)$. This further implies that ξ is essentially constant. If we denote by $c > 0$ the essential value of $|\xi|^2$, we obtain $c \cdot m_G(G) = \|\xi\|^2 < +\infty$ and so $m_G(G) < +\infty$. Then G is compact by Proposition 1.6. $\square$

The Koopman representation κ. Let G be any locally compact second countable group and $(X, \mathcal{B}, \nu)$ any standard probability space. We simply write (X, ν) in what follows. We endow G with its σ-algebra $\mathcal{B}(G)$ of Borel subsets. Let $G \curvearrowright (X, \nu)$ be any *probability measure preserving* (pmp) action meaning that the action map $G \times X \to X : (g, x) \mapsto gx$ is measurable (where we endow $G \times X$ with the product σ-algebra $\mathcal{B}(G) \otimes \mathcal{B}$) and that $g_*\nu = \nu$ for every $g \in G$. Denote by $\mathrm{L}^2(X, \nu)$ the Hilbert space of L^2-integrable functions on X. Since (X, ν) is a standard probability space, $\mathrm{L}^2(X, \nu)$ is separable (see e.g. [**Zi84**, Theorem A.11]). Define the *Koopman representation* $\kappa : G \to \mathcal{U}(\mathrm{L}^2(X, \nu))$ associated with the pmp action $G \curvearrowright (X, \nu)$ by the formula

$$\forall g \in G, \forall \xi \in \mathrm{L}^2(X, \nu), \quad (\kappa(g)\xi)(x) = \xi(g^{-1}x).$$

The Koopman representation $\kappa : G \to \mathcal{U}(\mathrm{L}^2(X, \nu))$ is a strongly continuous unitary representation. This follows from Lemma 2.2 after noticing that for all $\xi, \eta \in \mathrm{L}^2(X, \nu)$, the map

$$\varphi_{\xi,\eta} : G \to \mathbb{C} : g \mapsto \langle \kappa(g)\xi, \eta \rangle = \int_X \xi(g^{-1}x)\overline{\eta(x)} \, \mathrm{d}\nu(x)$$

is measurable thanks to Fubini's theorem. The constant function $\mathbf{1}_X$ is $\kappa(G)$-invariant. For this reason, it is natural to consider the restriction of the Koopman representation to the orthogonal complement $\mathrm{L}^2(X, \nu)^0 = \mathrm{L}^2(X, \nu) \ominus \mathbb{C}\mathbf{1}_X$ that we denote by $\kappa^0 : G \to \mathcal{U}(\mathrm{L}^2(X, \nu)^0)$.

We say that a measurable subset $Y \subset X$ is

- ν-a.e. G-invariant if for every $g \in G$, we have $\nu(gY \triangle Y) = 0$.
- strictly G-invariant if for every $g \in G$, we have $gY = Y$.

The next lemma clarifies the difference between the two notions.

LEMMA 2.6. *For any ν-a.e. G-invariant measurable subset $Y \subset X$, there is a strictly G-invariant measurable subset $Z \subset X$ such that $\nu(Y \triangle Z) = 0$.*

PROOF. Fix a left invariant Haar measure m_G on G. By assumption and using Fubini's theorem, the measurable subset

$$X_0 \doteq \{x \in X \mid G \to \mathbb{C} : g \mapsto \mathbf{1}_Y(g^{-1}x) \text{ is } m_G\text{-a.e. constant}\}$$

is ν-conull in X. For every $x \in X_0$, denote by $f(x)$ the unique essential value of the measurable function $G \to \mathbb{C} : g \mapsto \mathbf{1}_Y(g^{-1}x)$. For every $x \in X \setminus X_0$, set $f(x) = 0$. Note that $f(X) \subset \{0, 1\}$. Fubini's theorem implies that the function $f : X \to \mathbb{C}$ is measurable and $f(x) = \mathbf{1}_Y(x)$ for ν-almost every $x \in X$. For every $x \in X_0$ and every $h \in G$, the measurable function $G \to \mathbb{C} : g \mapsto \mathbf{1}_Y(g^{-1}h^{-1}x)$ is m_G-a.e. constant, hence $h^{-1}x \in X_0$ and $f(h^{-1}x) = f(x)$. This further implies that f is strictly G-invariant meaning that $f(g^{-1}x) = f(x)$ for every $g \in G$ and every $x \in X$. Set $Z \doteq \{x \in Z \mid f(x) = 1\}$. Then $Z \subset X$ is a strictly G-invariant measurable subset such that $\nu(Y \triangle Z) = 0$. $\square$

From now on, we simply say that the measurable subset $Y \subset X$ is G-invariant if for every $g \in G$, we have $\nu(gY \triangle Y) = 0$. We say that the pmp action $G \curvearrowright (X, \nu)$ is *ergodic* if every G-invariant measurable subset $Y \subset X$ is null or conull.

PROPOSITION 2.7. *Keep the same notation as above. Then $1_G \subset \kappa^0$ if and only if the pmp action $G \curvearrowright (X, \nu)$ is not ergodic.*

Proof. If the pmp action $G \curvearrowright (X,\nu)$ is not ergodic, then there exists a G-invariant measurable subset $Y \subset X$ such that $0 < \nu(Y) < 1$. Then the nonzero vector $\xi = \mathbf{1}_Y - \nu(Y)\mathbf{1}_X \in \mathrm{L}^2(X,\nu)^0$ is $\kappa^0(G)$-invariant. Conversely, assume that there exists a nonzero $\kappa^0(G)$-invariant vector $\xi \in \mathrm{L}^2(X,\nu)^0$. Up to taking the real or imaginary part of ξ, we may assume that ξ is real valued. Next, up to taking $\xi^+ = \max(\xi, 0)$ or $\xi^- = \max(-\xi, 0)$, we may further assume that $\xi \in \mathrm{L}^2(X,\nu)$ is $\kappa(G)$-invariant, nonnegative and $\xi \notin \mathbb{C}\mathbf{1}_X$. For every $t > 0$, define the G-invariant measurable subset $X_t = \{x \in X \mid \xi(x)^2 \geq t\}$. Then the function $\mathbb{R}_+^* \to \mathbb{R}_+ : t \mapsto \nu(X_t)$ is measurable, non-increasing and satisfies $\|\xi\|^2 = \int_0^{+\infty} \nu(X_t)\,\mathrm{d}t$. We claim that there exists $t > 0$ such that $0 < \nu(X_t) < 1$. Indeed otherwise there would exist $s > 0$ such that $\nu(X_t) = 0$ for every $t > s$ and $\nu(X_t) = 1$ for every $t \leq s$. This would imply that ξ is ν-almost everywhere constant equal to $\sqrt{s}$ and thus $\xi \in \mathbb{C}\mathbf{1}_X$, a contradiction. Therefore, there exists $t > 0$ such that $0 < \nu(X_t) < 1$. This shows that the pmp action $G \curvearrowright (X,\nu)$ is not ergodic. □

The quasi-regular representation $\lambda_{G/\Gamma}$. Let G be any locally compact second countable group and $\Gamma < G$ any lattice. We endow the locally compact second countable space $X = G/\Gamma$ with its σ-algebra $\mathcal{B}$ of Borel subsets (see Proposition 1.11(iii)). We denote by $\nu \in \mathrm{Prob}(X)$ the unique G-invariant Borel probability measure (see Proposition 1.15). Then the action $G \curvearrowright (X,\nu)$ is pmp. In that case, we denote by $\lambda_{G/\Gamma} : G \to \mathcal{U}(\mathrm{L}^2(G/\Gamma,\nu))$ the Koopman representation and we call it the *quasi-regular representation*. Since $G \curvearrowright X$ is transitive, Lemma 2.6 implies that $G \curvearrowright (X,\nu)$ is ergodic and Proposition 2.7 implies that $\lambda^0_{G/\Gamma} : G \to \mathcal{U}(\mathrm{L}^2(G/\Gamma,\nu)^0)$ is ergodic. We can strengthen the above result when $\Gamma < G$ is a *uniform* lattice.

Proposition 2.8. *Assume that $\Gamma < G$ is a uniform lattice. Then $\lambda^0_{G/\Gamma}$ has spectral gap.*

Proof. We may choose a Borel section $\sigma : X \to G$ such that $\sigma(X)$ is relatively compact in G (see Proposition 1.11 and Corollary 1.12). We further choose the Haar measure m_G on G such that $\sigma_*\nu = m_G|_{\sigma(X)}$. Set $Q = \sigma(X)\sigma(X)^{-1} \subset G$. Observe that $Q = Q^{-1}$ is relatively compact in G and so $m_G(Q) < +\infty$. Let $(\xi_n)_{n\in\mathbb{N}}$ be any bounded sequence of vectors in $\mathrm{L}^2(X,\nu)^0$ such that $\lim_n \sup_{g\in Q} \|\lambda^0_{G/\Gamma}(g)\xi_n - \xi_n\| = 0$. Using Fubini's theorem, we obtain

$$\begin{aligned}
\int_X |\xi_n(x)|^2\,\mathrm{d}\nu(x) &= \frac{1}{2}\int_X \left(\int_{\sigma(X)\sigma(x)^{-1}} |\xi_n(gx) - \xi_n(x)|^2\,\mathrm{d}m_G(g)\right)\mathrm{d}\nu(x) \\
&\leq \frac{1}{2}\int_X \left(\int_Q |\xi_n(gx) - \xi_n(x)|^2\,\mathrm{d}m_G(g)\right)\mathrm{d}\nu(x) \\
&= \frac{1}{2}\int_Q \left(\int_X |\xi_n(gx) - \xi_n(x)|^2\,\mathrm{d}\nu(x)\right)\mathrm{d}m_G(g)
\end{aligned}$$

$$= \frac{1}{2} \int_Q \|\lambda^0_{G/\Gamma}(g^{-1})\xi_n - \xi_n\|^2 \, \mathrm{d}m_G(g)$$
$$= \frac{1}{2} m_G(Q) \cdot \sup_{g \in Q} \|\lambda^0_{G/\Gamma}(g^{-1})\xi_n - \xi_n\| \to 0 \quad \text{as} \quad n \to +\infty.$$

This implies that $\lim_n \|\xi_n\| = 0$ and thus $\lambda^0_{G/\Gamma}$ has spectral gap. □

The previous result justifies to introduce the following terminology.

Definition 2.9. Let G be any locally compact second countable group. We say that a lattice $\Gamma < G$ is *weakly uniform* if $\lambda^0_{G/\Gamma}$ has spectral gap.

Proposition 2.8 shows that any uniform lattice $\Gamma < G$ is weakly uniform. In Section 3, we will provide examples of nonuniform weakly uniform lattices $\Gamma < G$.

1.2. Induction of unitary representations. Whenever $\Gamma < G$ is a lattice, we explain how to naturally associate to any unitary representation $\pi : \Gamma \to \mathcal{U}(\mathcal{H}_\pi)$ a strongly continuous unitary representation $\widehat{\pi} : G \to \mathcal{U}(\mathcal{H}_{\widehat{\pi}})$. We then compute the *induced* representation $\widehat{\pi}$ in several examples.

Let G be any locally compact second countable group and $\Gamma < G$ any lattice. Set $X = G/\Gamma$ and denote by $\nu \in \mathrm{Prob}(X)$ the unique G-invariant (regular) Borel probability measure on X. Choose a Borel section $\sigma : X \to G$ as in Corollary 1.12. Define the Borel map $\tau : G \times X \to \Gamma : (g, x) \mapsto \sigma(gx)^{-1} g \sigma(x)$. Observe that for every $g \in G$ and every $x \in X$, $\tau(g, x) \in \Gamma$ is the unique element $\gamma \in \Gamma$ such that $g\sigma(x) = \sigma(gx)\, \tau(g, x)$. The Borel map τ satisfies the 1-*cocycle relation*

$$(2.1) \qquad \forall g_1, g_2 \in G, \forall x \in X, \quad \tau(g_1 g_2, x) = \tau(g_1, g_2 x)\, \tau(g_2, x).$$

We present two different viewpoints to define the *induction* from Γ to G for unitary representations. Let $\pi : \Gamma \to \mathcal{U}(\mathcal{H}_\pi)$ by any unitary representation. Assume that $\mathcal{H}_\pi$ is separable so that $\mathcal{H}_\pi$ is a Polish space.

Induction I. Denote by $\mathrm{L}^2(X, \nu, \mathcal{H}_\pi)$ the Hilbert space of ν-equivalence classes of all measurable functions $\eta : X \to \mathcal{H}_\pi$ that satisfy

- $\int_X \|\eta(x)\|^2 \, \mathrm{d}\nu(x) < +\infty$.

Endowed with the sesquilinear form defined by

$$\forall \eta_1, \eta_2 \in \mathrm{L}^2(X, \nu, \mathcal{H}_\pi), \quad \langle \eta_1, \eta_2 \rangle_\nu = \int_X \langle \eta_1(x), \eta_2(x) \rangle \, \mathrm{d}\nu(x),$$

the space $\mathrm{L}^2(X, \nu, \mathcal{H}_\pi)$ is indeed a Hilbert space. We may and will identify $\mathrm{L}^2(X, \nu, \mathcal{H}_\pi)$ with the tensor product Hilbert space $\mathrm{L}^2(X, \nu) \otimes \mathcal{H}_\pi$. We refer the reader to [**BHV08**, Appendix E] for further details.

Definition 2.10. With this viewpoint, the *induced representation* $\widehat{\pi}_1 : G \to \mathcal{U}(\mathrm{L}^2(X, \nu, \mathcal{H}_\pi))$ is defined by the formula

$$\forall g \in G, \forall \eta \in \mathrm{L}^2(X, \nu, \mathcal{H}_\pi), \quad (\widehat{\pi}_1(g)\eta)(x) = \pi(\tau(g, g^{-1}x))\eta(g^{-1}x).$$

Simply write $\mathcal{H}_{\widehat{\pi}} = \mathrm{L}^2(X, \nu, \mathcal{H}_\pi)$. The induced representation $\widehat{\pi}_1 : G \to \mathcal{U}(\mathcal{H}_{\widehat{\pi}})$ is a strongly continuous unitary representation. This follows from Lemma 2.2 after noticing that for all $\eta_1, \eta_2 \in \mathcal{H}_{\widehat{\pi}}$, the map

$$\varphi_{\eta_1,\eta_2} : G \to \mathbb{C} : g \mapsto \int_X \langle \pi(\tau(g, g^{-1}x))\eta_1(g^{-1}x), \eta_2(x)\rangle \,\mathrm{d}\nu(x)$$

is measurable thanks to Fubini's theorem.

Induction II. Denote by $\mathcal{F} = \sigma(X) \subset G$ and recall that $\mathcal{F} \subset G$ is a Borel fundamental domain for the right translation action $\Gamma \curvearrowright G$ and so $G = \mathcal{F} \cdot \Gamma$. Denote by m_G the unique Haar measure on G such that $\sigma_*\nu = m_G|_{\mathcal{F}}$. Denote by $\mathrm{L}^2(G, \mathcal{H}_\pi)^\Gamma$ the Hilbert space of m_G-equivalence classes of all measurable functions $\xi : G \to \mathcal{H}_\pi$ that satisfy

- For m_G-almost every $g \in G$ and every $\gamma \in \Gamma$, $\xi(g\gamma^{-1}) = \pi(\gamma)\xi(g)$.
- $\int_{\mathcal{F}} \|\xi(g)\|^2 \,\mathrm{d}m_G(g) < +\infty$.

Endowed with the sesquilinear form defined by

$$\forall \xi_1, \xi_2 \in \mathrm{L}^2(G, \mathcal{H}_\pi)^\Gamma, \quad \langle \xi_1, \xi_2\rangle = \int_{\mathcal{F}} \langle \xi_1(g), \xi_2(g)\rangle \,\mathrm{d}m_G(g),$$

the space $\mathrm{L}^2(G, \mathcal{H}_\pi)^\Gamma$ is indeed a Hilbert space.

Definition 2.11. With this view point, the *induced representation* $\widehat{\pi}_2 : G \to \mathcal{U}(\mathrm{L}^2(G, \mathcal{H}_\pi)^\Gamma)$ is defined by the formula

$$\forall g \in G, \forall \eta \in \mathrm{L}^2(G, \mathcal{H}_\pi)^\Gamma, \quad (\widehat{\pi}_2(g)\xi)(h) = \xi(g^{-1}h).$$

Let us explain why this second view point is actually equivalent to the first viewpoint. Define the mapping $U : \mathrm{L}^2(G, \mathcal{H}_\pi)^\Gamma \to \mathcal{H}_{\widehat{\pi}}$ by the formula $(U\xi)(x) = \xi(\sigma(x))$ for all $\xi \in \mathrm{L}^2(G, \mathcal{H}_\pi)^\Gamma$. Then it is plain to see that U is a unitary operator such that $U^* : \mathcal{H}_{\widehat{\pi}} \to \mathrm{L}^2(G, \mathcal{H}_\pi)^\Gamma$ is given by the formula $(U^*\eta)(g) = \pi(\tau(g^{-1}, g\Gamma))\eta(g\Gamma)$ for all $\eta \in \mathcal{H}_{\widehat{\pi}}$. Moreover, for every $g \in G$, we have $\widehat{\pi}_2(g) = U^*\widehat{\pi}_1(g)U$. Therefore, $\widehat{\pi}_1$ and $\widehat{\pi}_2$ are unitarily equivalent.

In what follows, it will be useful to switch from one viewpoint to the other. We will simply denote by $\widehat{\pi} : G \to \mathcal{U}(\mathcal{H}_{\widehat{\pi}})$ the induced representation and we will emphasize (when necessary) the viewpoint we choose.

Examples 2.12. Keep the same notation as above. The following verifications are left as an exercise.

(i) Assume that $\pi = 1_\Gamma$ is the trivial representation. Then $\widehat{\pi} = \lambda_{G/\Gamma}$ is the quasi-regular representation.
(ii) Assume that $\pi = \lambda_\Gamma$ is the left regular representation for Γ. Then $\widehat{\pi} = \lambda_G$ is the left regular representation for G.
(iii) Let $\pi : G \to \mathcal{U}(\mathcal{H}_\pi)$ be any strongly continuous unitary representation and let $\rho = \pi|_\Gamma : \Gamma \to \mathcal{U}(\mathcal{H}_\pi)$ be the restriction. Then $\widehat{\rho}$ and $\pi \otimes \lambda_{G/\Gamma}$ are unitarily equivalent.

The following result will turn out to be useful later on.

Proposition 2.13. *Let $\pi : \Gamma \to \mathcal{U}(\mathcal{H}_\pi)$ be any unitary representation and denote by $\widehat{\pi} : G \to \mathcal{U}(\mathcal{H}_{\widehat{\pi}})$ the induced representation. Then $1_\Gamma \subset \pi$ if and only if $1_G \subset \widehat{\pi}$.*

Proof. Choose the first viewpoint on induction. Keep the same notation as above. Firstly, assume that $1_\Gamma \subset \pi$ and choose a $\pi(\Gamma)$-invariant unit vector $\xi \in \mathcal{H}_\pi$. Then $\eta = \mathbf{1}_X \otimes \xi \in \mathcal{H}_{\widehat{\pi}}$ is a $\widehat{\pi}(G)$-invariant unit vector and so $1_G \subset \widehat{\pi}$.

Conversely, assume that $1_G \subset \widehat{\pi}$ and choose a $\widehat{\pi}(G)$-invariant unit vector $\eta \in \mathcal{H}_{\widehat{\pi}}$. Then for every $g \in G$ and ν-almost every $x \in X$, we have

$$\pi(\tau(g, g^{-1}x))\eta(g^{-1}x) = \eta(x).$$

For every $x \in X$, define the measurable function $\zeta_x : G \to \mathcal{H}_\pi : g \mapsto \pi(\tau(g, g^{-1}x))\eta(g^{-1}x)$. Set

$$Y \doteq \{x \in X \mid \zeta_x \text{ is } m_G\text{-a.e. constant}\}.$$

For every $x \in Y$, denote by $\zeta(x) \in \mathcal{H}_\pi$ the unique essential value of the function ζ_x. Fubini's theorem implies that $Y \subset X$ is conull, the function $\zeta : Y \to \mathcal{H}_\pi : x \mapsto \zeta(x)$ is measurable and $\zeta(x) = \eta(x)$ for ν-almost every $x \in Y$. Viewing $\zeta \in \mathcal{H}_{\widehat{\pi}}$, we have $\zeta = \eta$.

Claim 2.14. For every $x \in Y$ and every $h \in G$, we have $h^{-1}x \in Y$ and $\pi(\tau(h, h^{-1}x))\zeta(h^{-1}x) = \zeta(x)$.

Indeed, using the cocycle relation (2.1) and since $x \in Y$, for m_G-almost every $g \in G$, we have

$$\begin{aligned}\pi(\tau(h, h^{-1}x))^* \zeta(x) &= \pi(\tau(h, h^{-1}x))^* \pi(\tau(hg, g^{-1}h^{-1}x))\eta(g^{-1}h^{-1}x) \\ &= \pi(\tau(g, g^{-1}h^{-1}x))\eta(g^{-1}h^{-1}x).\end{aligned}$$

Since the left hand side does not depend on $g \in G$, this implies that $h^{-1}x \in Y$ and $\pi(\tau(h, h^{-1}x))\zeta(h^{-1}x) = \zeta(x)$. This finishes the proof of Claim 2.14.

Claim 2.14 implies that $Y \subset X$ is G-invariant and so $Y = X = G/\Gamma$. Moreover, for every $x \in X$ and every $h \in G$, we have $\pi(\tau(h, h^{-1}x))\zeta(h^{-1}x) = \zeta(x)$. Set $\xi = \zeta(\Gamma) \in \mathcal{H}_\pi$. We have $\xi \neq 0$, otherwise we would have $\zeta(h^{-1}\Gamma) = \pi(\tau(h, h^{-1}\Gamma))^*\zeta(\Gamma) = 0$ for every $h \in G$. This would imply that $\eta = \zeta = 0$, a contradiction. Then $\xi \neq 0$ and for every $\gamma \in \Gamma$, we have $\pi(\gamma)\xi = \pi(\tau(\gamma, \gamma^{-1}\Gamma))\zeta(\gamma^{-1}\Gamma) = \zeta(\Gamma) = \xi$. Then $\xi \in \mathcal{H}_\pi$ is a nonzero $\pi(\Gamma)$-invariant vector and so $1_\Gamma \subset \pi$. □

2. Amenability

Definition 2.15. Let G be any locally compact group. We say that G is *amenable* if any affine continuous action $G \curvearrowright \mathcal{C}$ on a nonempty convex compact subset of a locally convex topological vector space has a G-fixed point.

We give a few examples of locally compact amenable groups.

Proposition 2.16. *Any compact group is amenable.*

Proof. Denote by m_G the (unique) Haar probability measure on G. Let $G \curvearrowright \mathcal{C}$ be any affine continuous action on a nonempty convex compact subset of a locally convex topological vector space. Define the convex weak*-compact subset $\mathrm{Prob}(\mathcal{C}) = \{\mu \in \mathrm{C}_{\mathbb{R}}(\mathcal{C})^* \mid \mu \geq 0 \text{ and } \mu(\mathbf{1}_{\mathcal{C}}) = 1\}$ and consider the affine weak*-continuous action $G \curvearrowright \mathrm{Prob}(\mathcal{C})$ defined by

$$\forall g \in G, \forall f \in \mathrm{C}_{\mathbb{R}}(\mathcal{C}), \forall \mu \in \mathrm{Prob}(\mathcal{C}), \quad (g_*\mu)(f) = \mu(f \circ g).$$

Define the *barycenter* map $\mathrm{Bar} : \mathrm{Prob}(\mathcal{C}) \to \mathcal{C}$ as the unique continuous map satisfying $f(\mathrm{Bar}(\mu)) = \mu(f)$ for every real-valued continuous affine function $f \in \mathcal{A}_{\mathbb{R}}(\mathcal{C})$. Since $G \curvearrowright \mathcal{C}$ is continuous affine, $\mathrm{Bar} : \mathrm{Prob}(\mathcal{C}) \to \mathcal{C}$ is G-equivariant. Choose a point $c \in \mathcal{C}$ and define the G-equivariant continuous orbital map $\iota : G \to \mathcal{C} : g \mapsto gc$. We may define $\mu = \iota_* m_G \in \mathrm{Prob}(\mathcal{C})$. Since m_G is a left invariant Borel measure, it follows that $g_*\mu = \mu$ for every $g \in G$. This further implies that $\mathrm{Bar}(\mu) \in \mathcal{C}$ is a G-fixed point. □

Proposition 2.17. *Any abelian locally compact group is amenable.*

Proof. Let $G \curvearrowright \mathcal{C}$ be any continuous affine action on a nonempty convex compact subset of a locally convex topological vector space. Whenever $\mathcal{F} \subset G$ is a finite subset, denote by $\mathcal{C}^{\mathcal{F}}$ the convex compact subset of $\mathcal{F}$-fixed points in $\mathcal{C}$. Since G is abelian, G leaves $\mathcal{C}^{\mathcal{F}}$ globally invariant. If we show that the compact subset $\mathcal{C}^{\mathcal{F}}$ is nonempty for every finite subset $\mathcal{F} \subset G$, by finite intersection property, we will have that the compact subset of G-fixed points $\mathcal{C}^G = \bigcap\{\mathcal{C}^{\mathcal{F}} \mid \mathcal{F} \subset G \text{ finite subset}\}$ is nonempty. It remains to prove that $\mathcal{C}^{\mathcal{F}}$ is nonempty for every finite subset $\mathcal{F} \subset G$. By induction and since G is abelian, it suffices to prove that $\mathcal{C}^g = \{c \in \mathcal{C} \mid gc = c\}$ is nonempty for every $g \in G$. This in turn follows from Markov–Kakutani's fixed point theorem. Choose $c \in \mathcal{C}$ and for every $n \in \mathbb{N}$, set

$$c_n = \frac{1}{n+1}(c + gc + \cdots + g^n c) \in \mathcal{C}.$$

By compactness, denote by $c_\infty \in \mathcal{C}$ an accumulation point of the sequence $(c_n)_{n\in\mathbb{N}}$. Since $\frac{1}{n+2}c + \frac{n+1}{n+2}gc_n = \frac{n+1}{n+2}c_n + \frac{1}{n+2}g^{n+1}c$ and since g is a homeomorphism of $\mathcal{C}$, it follows that $gc_\infty = c_\infty$ and so $c_\infty \in \mathcal{C}^g$. □

We prove various permanence properties enjoyed by amenable locally compact groups.

Proposition 2.18. *Let G, H be any locally compact groups. Assume that G is amenable. The following assertions hold:*

(i) *If $\rho : G \to H$ is a continuous homomorphism with dense range, then H is amenable.*
(ii) *If $H \lhd G$ is a closed normal subgroup, then G/H is amenable.*

Proof. (i) Let $H \curvearrowright \mathcal{C}$ be any continuous affine action on a nonempty convex compact subset of a locally convex topological vector space. By composing with $\rho : G \to H$, we obtain a continuous affine G-action. Since G is amenable, the continuous affine G-action has a G-fixed point. This shows that the original continuous affine H-action has a $\rho(G)$-fixed point. By continuity and density of $\rho(G)$ in H, we obtain a H-fixed point. Thus, $H = \overline{\rho(G)}$ is amenable.

(ii) It suffices to apply item (i) to the continuous homomorphism $G \to G/H$. $\square$

Let now G be any locally compact σ-compact group. As usual, we denote by $\mathcal{B}(G)$ the σ-algebra of Borel subsets of G and we fix a left invariant Haar measure m_G on G. Denote by $\Delta_G : G \to \mathbb{R}_+^*$ the modular function. For every $p \in [1, +\infty]$, we simply write $\mathrm{L}^p(G) = \mathrm{L}^p(G, \mathcal{B}(G), m_G)$. Since G is σ-compact, m_G is σ-finite and hence we have $\mathrm{L}^\infty(G) = \mathrm{L}^1(G)^*$. We denote by $\lambda : G \curvearrowright \mathrm{L}^p(G)$ the *left translation action* defined by

$$\forall g \in G, \forall F \in \mathrm{L}^p(G), \quad (\lambda(g)F)(h) = F(g^{-1}h).$$

The left translation action $\lambda : G \curvearrowright \mathrm{L}^p(G)$ is isometric for every $p \in [1, +\infty]$ and continuous for every $p \in [1, +\infty)$. Since $G \curvearrowright \mathrm{L}^\infty(G)$ need not be continuous, we denote by $\mathrm{UC}_\ell(G) \subset \mathrm{L}^\infty(G)$ the subspace of *left uniformly continuous* functions

$$\mathrm{UC}_\ell(G) \doteq \{F \in \mathrm{L}^\infty(G) \mid \|\lambda(g)F - F\|_\infty \to 0 \text{ as } g \to e\}.$$

Observe that $\mathrm{UC}_\ell(G) \subset \mathrm{L}^\infty(G)$ is a $\lambda(G)$-invariant $\|\cdot\|_\infty$-closed subspace. Letting $\mathrm{C}_b(G)$ be the space of bounded continuous functions on G, we have the following inclusions $\mathrm{UC}_\ell(G) \subset \mathrm{C}_b(G) \subset \mathrm{L}^\infty(G)$. Observe that when G is discrete, we have $\mathrm{UC}_\ell(G) = \mathrm{C}_b(G) = \ell^\infty(G)$. Whenever $\mathcal{F} \subset \mathrm{L}^\infty(G)$ is a $\|\cdot\|_\infty$-closed subspace such that $\mathbb{C}\mathbf{1}_G \subset \mathcal{F}$, we say that an element $\mathfrak{m} \in \mathcal{F}^*$ is a *mean* if $\mathfrak{m}(F) \geq 0$ for every $F \in \mathcal{F}_+$ and $\mathfrak{m}(\mathbf{1}_G) = 1$. If $\mathcal{F} \subset \mathrm{L}^\infty(G)$ is moreover $\lambda(G)$-invariant, we say that $\mathfrak{m} \in \mathcal{F}^*$ is a *left invariant mean* if $\mathfrak{m}(\lambda(g)F) = \mathfrak{m}(F)$ for every $g \in G$ and every $F \in \mathcal{F}$.

Recall that the *convolution product* of two measurable functions $F_1, F_2 : G \to \mathbb{C}$, whenever it makes sense, is defined as

$$(F_1 * F_2)(h) = \int_G F_1(g)F_2(g^{-1}h)\,\mathrm{d}m_G(g).$$

Set $\mathrm{P}(G) = \{\mu \in \mathrm{L}^1(G) \mid \mu \geq 0 \text{ and } \|\mu\|_1 = 1\}$. We will use the following technical lemma whose proof is left to the reader.

Lemma 2.19. *The following assertions hold:*

(i) *If $\mu \in \mathrm{P}(G)$ and $F \in \mathrm{L}^\infty(G)$, then $\mu * F \in \mathrm{UC}_\ell(G)$.*
(ii) *If $(\mu_i)_{i\in I}$ is a net in $\mathrm{L}^1(G)$ such that $\lim_i \|\mu_i\|_1 = 0$, then for every $F \in \mathrm{L}^\infty(G)$, we have $\lim_i \|\mu_i * F\|_\infty = 0$.*
(iii) *There exists a net $(\mu_i)_{i\in I}$ in $\mathrm{P}(G)$ such that for every $\mu \in \mathrm{L}^1(G)$, we have $\lim_i \|\mu_i * \mu - \mu\|_1 = \lim_i \|\mu * \mu_i - \mu\|_1 = 0$.*
(iv) *If $g \in G$, $\mu \in \mathrm{P}(G)$ and $F \in \mathrm{L}^\infty(G)$, then $(\lambda(g)\mu)*F = \lambda(g)(\mu*F)$.*

The main result of this section is a functional analytic characterization of amenability for locally compact groups.

THEOREM 2.20. *Let G be any locally compact σ-compact group. The following conditions are equivalent:*

(i) $1_G \prec \lambda_G$, *that is, the left regular representation λ_G has almost invariant vectors.*
(ii) *There exists a left invariant mean $\mathfrak{m} \in \mathrm{L}^\infty(G)^*$.*
(iii) *There exists a left invariant mean $\mathfrak{m} \in \mathrm{UC}_\ell(G)^*$.*
(iv) *G is amenable, that is, any affine continuous action $G \curvearrowright \mathcal{C}$ on a nonempty convex compact subset of a locally convex topological vector space has a G-fixed point.*

PROOF. (i) $\Rightarrow$ (ii) There exists a net $(\xi_i)_{i\in I}$ of unit vectors in $\mathrm{L}^2(G)$ such that for every compact subset $Q \subset G$, we have

$$\lim_i \sup_{g\in Q} \|\lambda_G(g)\xi_i - \xi_i\|_2 = 0.$$

Choose a nonprincipal ultrafilter $\mathcal{U}$ on I. Define the unital $*$-homomorphism $\rho : \mathrm{L}^\infty(G) \to \mathbb{B}(\mathrm{L}^2(G))$ by the formula $\rho(F)\xi = F\xi$ for every $F \in \mathrm{L}^\infty(G)$ and every $\xi \in \mathrm{L}^2(G)$. Then we have $\lambda_G(g)\rho(F)\lambda_G(g)^* = \rho(\lambda(g)F)$ for every $g \in G$ and every $F \in \mathrm{L}^\infty(G)$. Define the mean $\mathfrak{m} \in \mathrm{L}^\infty(G)^*$ by the formula

$$\forall F \in \mathrm{L}^\infty(G), \quad \mathfrak{m}(F) \doteq \lim_{i\to\mathcal{U}} \langle \rho(F)\xi_i, \xi_i\rangle.$$

Then for every $g \in G$ and every $F \in \mathrm{L}^\infty(G)$, we have

$$\begin{aligned}
\mathfrak{m}(\lambda(g)F) &= \lim_{i\to\mathcal{U}} \langle \rho(\lambda(g)F)\xi_i, \xi_i\rangle \\
&= \lim_{i\to\mathcal{U}} \langle \lambda_G(g)\rho(F)\lambda_G(g)^*\xi_i, \xi_i\rangle \\
&= \lim_{i\to\mathcal{U}} \langle \rho(F)\lambda_G(g)^*\xi_i, \lambda_G(g)^*\xi_i\rangle \\
&= \mathfrak{m}(F).
\end{aligned}$$

Thus, $\mathfrak{m} \in \mathrm{L}^\infty(G)^*$ is a left invariant mean.

(ii) $\Rightarrow$ (iii) This is trivial.

(iii) $\Rightarrow$ (iv) As in Proposition 2.16, define the convex weak*-compact subset $\mathrm{Prob}(\mathcal{C}) = \{\mu \in \mathrm{C}_\mathbb{R}(\mathcal{C})^* \mid \mu \geq 0 \text{ and } \mu(\mathbf{1}_\mathcal{C}) = 1\}$ and consider the affine weak*-continuous action $G \curvearrowright \mathrm{Prob}(\mathcal{C})$ defined by

$$\forall g \in G, \forall f \in \mathrm{C}_\mathbb{R}(\mathcal{C}), \forall \mu \in \mathrm{Prob}(\mathcal{C}), \quad (g_*\mu)(f) = \mu(f \circ g).$$

Recall that the barycenter map $\mathrm{Bar} : \mathrm{Prob}(\mathcal{C}) \to \mathcal{C}$ is the unique continuous map satisfying $f(\mathrm{Bar}(\mu)) = \mu(f)$ for every real-valued continuous affine function $f \in \mathcal{A}_\mathbb{R}(\mathcal{C})$. Since $G \curvearrowright \mathcal{C}$ is continuous affine, $\mathrm{Bar} : \mathrm{Prob}(\mathcal{C}) \to \mathcal{C}$ is G-equivariant. Choose a point $c \in \mathcal{C}$ and define the G-equivariant continuous orbital map $\iota : G \to \mathcal{C} : g \mapsto gc$. For every $f \in \mathrm{C}_\mathbb{R}(\mathcal{C})$, we have $f \circ \iota \in \mathrm{UC}_\ell(G)$. We may define $\mu \in \mathrm{Prob}(\mathcal{C})$ by the formula

$$\forall f \in \mathrm{C}_\mathbb{R}(\mathcal{C}), \quad \mu(f) = \mathfrak{m}(f \circ \iota).$$

Since $\mathfrak{m} \in \mathrm{UC}_\ell(G)^*$ is a left invariant mean, it follows that $g_*\mu = \mu$ for every $g \in G$. This further implies that $\mathrm{Bar}(\mu) \in \mathcal{C}$ is a G-fixed point.

(iv) $\Rightarrow$ (iii) Endow $E = \mathrm{UC}_\ell(G)^*$ with the weak*-topology and consider the nonempty convex weak*-compact subset $\mathcal{C} \subset \mathrm{UC}_\ell(G)^*$ of all means on $\mathrm{UC}_\ell(G)$. Since the action $G \curvearrowright \mathrm{UC}_\ell(G)$ is $\|\cdot\|_\infty$-continuous, the action $G \curvearrowright \mathcal{C}$ is affine weak*-continuous. Thus, there exists a G-fixed point $\mathfrak{m} \in \mathcal{C}$ and so $\mathfrak{m} \in \mathrm{UC}_\ell(G)^*$ is a left invariant mean.

(iii) $\Rightarrow$ (i) We proceed in several intermediate steps. Let $\mathfrak{m} \in \mathrm{UC}_\ell(G)^*$ be a left invariant mean.

Claim 2.21. For every $\mu \in \mathrm{P}(G)$ and every $F \in \mathrm{UC}_\ell(G)$, we have $\mathfrak{m}(\mu * F) = \mathfrak{m}(F)$.

Indeed, let $\mu \in \mathrm{P}(G)$ and $F \in \mathrm{UC}_\ell(G)$. Observe that using Lemma 2.19(ii), we may assume that $\mu \in \mathrm{P}(G)$ is compactly supported. Then denote by $K = \mathrm{supp}(\mu) \subset G$ the compact support of $\mu \in \mathrm{P}(G)$. The G-equivariant mapping $\iota : G \to \mathrm{UC}_\ell(G) : g \mapsto \lambda(g)F$ is continuous and thus $\iota(K) \subset \mathrm{UC}_\ell(G)$ is a compact subset. Then the closed convex hull $\mathcal{C}$ of $\iota(K)$ is a convex compact subset of $\mathrm{UC}_\ell(G)$ (see [**Ru91**, Theorem 3.20]). Set $\nu = \iota_*\mu$ and regard $\nu \in \mathrm{Prob}(\mathcal{C})$ by the formula

$$\forall f \in \mathrm{C}_{\mathbb{R}}(\mathcal{C}), \quad \nu(f) = \int_G \mu(g) f(\lambda(g)F)\, \mathrm{d}m_G(g).$$

We claim that $\mu * F = \mathrm{Bar}(\nu) \in \mathcal{C}$. Recall that $f(\mathrm{Bar}(\nu)) = \nu(f)$ for every $f \in \mathcal{A}_{\mathbb{R}}(\mathcal{C})$. For every $h \in G$, regarding the evaluation map $e_h : \mathrm{UC}_\ell(G) \to \mathbb{C} : f \mapsto f(h)$ as an element of $\mathcal{A}_{\mathbb{R}}(\mathcal{C})$, we have

$$\mathrm{Bar}(\nu)(h) = e_h(\mathrm{Bar}(\nu)) = \nu(e_h) = \int_G \mu(g) e_h(\lambda(g)F)\, \mathrm{d}m_G(g) = (\mu * F)(h).$$

Thus, we have $\mathrm{Bar}(\nu) = \mu * F$. Since $\mathfrak{m} \in \mathrm{UC}_\ell(G)^*$ is a left invariant mean, we can regard $\mathfrak{m} \in \mathcal{A}_{\mathbb{R}}(\mathcal{C})$ and we obtain

$$\mathfrak{m}(\mu * F) = \mathfrak{m}(\mathrm{Bar}(\nu)) = \int_G \mu(g)\mathfrak{m}(\lambda(g)F)\, \mathrm{d}m_G(g) = \mathfrak{m}(F).$$

This finishes the proof of Claim 2.21.

Claim 2.22. There exists a mean $\mathfrak{m}_0 \in \mathrm{L}^\infty(G)^*$ such that for every $\mu \in \mathrm{P}(G)$ and every $F \in \mathrm{L}^\infty(G)$, we have $\mathfrak{m}_0(\mu * F) = \mathfrak{m}_0(F)$.

Indeed, choose any $\mu_0 \in \mathrm{P}(G)$. Thanks to Lemma 2.19(i), we may define the mean $\mathfrak{m}_0 \in \mathrm{L}^\infty(G)^*$ by the formula $\mathfrak{m}_0(F) = \mathfrak{m}(\mu_0 * F)$ for every $F \in \mathrm{L}^\infty(G)$. Choose a net as in Lemma 2.19(iii). Using Lemma 2.19(ii), for every $\mu \in \mathrm{P}(G)$, we have

$$\begin{aligned}
\mathfrak{m}_0(\mu * F) &= \lim_i \mathfrak{m}_0(\mu * \mu_i * F) \\
&= \lim_i \mathfrak{m}(\mu_0 * \mu * \mu_i * F) \\
&= \lim_i \mathfrak{m}(\mu_i * F) \quad \text{by Claim 2.21} \\
&= \lim_i \mathfrak{m}(\mu_0 * \mu_i * F) \quad \text{by Claim 2.21}
\end{aligned}$$

$$= \mathfrak{m}(\mu_0 * F)$$
$$= \mathfrak{m}_0(F).$$

This finishes the proof of Claim 2.22.

Denote by $\mathcal{M}$ the nonempty convex weak*-compact subset of all means on $\mathrm{L}^\infty(G)$. Hahn–Banach theorem implies that the map $\mathrm{P}(G) \to \mathcal{M} : \mu \mapsto \mathfrak{m}_\mu$ defined by the formula $\mathfrak{m}_\mu(F) = \int_G \mu(g)F(g)\,\mathrm{d}m_G(g)$ for every $F \in \mathrm{L}^\infty(G)$ has dense range. Thus, we can find a net $(\mu_i)_{i\in I}$ in $\mathrm{P}(G)$ such that $\mathfrak{m}_{\mu_i} \to \mathfrak{m}_0$ for the weak*-topology. For every $\mu \in \mathrm{P}(G)$, define $\mu^{\mathrm{op}} \in \mathrm{P}(G)$ by the formula $\mu^{\mathrm{op}}(g) = \Delta_G(g)^{-1}\mu(g^{-1})$. For every $\mu \in \mathrm{P}(G)$ and every $F \in \mathrm{L}^\infty(G)$, using Fubini's theorem, we have

$$\begin{aligned}
\int_G (\mu * \mu_i)(g)F(g)\,\mathrm{d}m_G(g) &= \int_{G\times G} \mu(h)\mu_i(h^{-1}g)\,F(g)\,\mathrm{d}m_G^{\otimes 2}(g,h)\\
&= \int_{G\times G} \mu_i(h^{-1}g)\,\mu(h)F(g)\,\mathrm{d}m_G^{\otimes 2}(g,h)\\
&= \int_{G\times G} \mu_i(g)\,\mu(h)F(hg)\,\mathrm{d}m_G^{\otimes 2}(g,h)\\
&= \int_{G\times G} \mu_i(g)\,\mu^{\mathrm{op}}(h)F(h^{-1}g)\,\mathrm{d}m_G^{\otimes 2}(g,h)\\
&= \int_{G\times G} \mu_i(g)\,(\mu^{\mathrm{op}} * F)(g)\,\mathrm{d}m_G(g).
\end{aligned}$$

Then Claim 2.22 implies that for every $\mu \in \mathrm{P}(G)$, $\mu * \mu_i - \mu_i \to 0$ weakly in $\mathrm{L}^1(G)$. Denote by J the directed set of all pairs $(\varepsilon, \mathcal{F})$ where $\varepsilon > 0$ and $\mathcal{F} \subset \mathrm{P}(G)$ is a finite subset endowed with the order $(\varepsilon_1, \mathcal{F}_1) \leq (\varepsilon_2, \mathcal{F}_2)$ if and only if $\varepsilon_1 \leq \varepsilon_2$ and $\mathcal{F}_2 \subset \mathcal{F}_1$. Let $j = (\varepsilon, \mathcal{F}) \in J$ and consider the Banach space $(E_j, \|\cdot\|) = \bigoplus_{\mu\in\mathcal{F}}(\mathrm{L}^1(G), \|\cdot\|_1)$. The weak topology on E_j is simply the product of the weak topologies on $\mathrm{L}^1(G)$. Then 0 belongs to the weak closure in E_j of the convex subset

$$\mathcal{C}_j \doteq \{(\mu * \psi - \psi)_{\mu\in\mathcal{F}} \mid \psi \in \mathrm{P}(G)\} \subset E_j.$$

Hahn–Banach theorem implies that 0 belongs to the strong closure in E_j of $\mathcal{C}_j$. Then we may find $\psi_j \in \mathrm{P}(G)$ such that for every $\mu \in \mathcal{F}$, we have $\|\mu * \psi_j - \psi_j\|_1 < \varepsilon$. Thus, we have found a net $(\psi_j)_{j\in J}$ in $\mathrm{P}(G)$ such that for every $\mu \in \mathrm{P}(G)$, we have $\lim_j \|\mu * \psi_j - \psi_j\|_1 = 0$.

Note that for every nonempty $\|\cdot\|_1$-compact subset $K \subset \mathrm{P}(G)$, we have $\lim_j \|\mu * \psi_j - \psi_j\|_1 = 0$ *uniformly* on K. Indeed, let $\varepsilon > 0$ and choose $\mu_1, \dots, \mu_n \in K$ such that for every $\mu \in K$, there exists $1 \leq i \leq n$ for which $\|\mu - \mu_i\| \leq \varepsilon$. Choose $j_0 \in J$ such that $\|\mu_i * \psi_j - \psi_j\|_1 \leq \varepsilon$ for every $1 \leq i \leq n$ and every $j \geq j_0$. Then for every $\mu \in K$ and every $j \geq j_0$, choosing $1 \leq i \leq n$ such that $\|\mu - \mu_i\| \leq \varepsilon$, we have

$$\begin{aligned}
\|\mu * \psi_j - \psi_j\|_1 &\leq \|(\mu - \mu_i) * \psi_j\|_1 + \|\mu_i * \psi_j - \psi_j\|_1\\
&\leq \|\mu - \mu_i\|_1 + \|\mu_i * \psi_j - \psi_j\|_1
\end{aligned}$$

$$\leq 2\varepsilon.$$

This shows that $\lim_j \|\mu * \psi_j - \psi_j\|_1 = 0$ *uniformly* on K.

Fix $\varepsilon > 0$ and $Q \subset G$ any compact subset. Fix any $\mu \in \mathrm{P}(G)$. The orbital map $G \to \mathrm{P}(G) : g \mapsto \lambda(g)\mu$ is $\|\cdot\|_1$-continuous and so $\iota(Q) \subset \mathrm{P}(G)$ is $\|\cdot\|_1$-compact. Lemma 2.19(iv) implies that

$$\sup_{g\in Q} \|\lambda(g)(\mu * \psi_j) - \mu * \psi_j\|_1 = \sup_{g \in Q} \|(\lambda(g)\mu) * \psi_j - \mu * \psi_j\|_1 \to 0.$$

We may find $j \in J$ large enough so that with $\zeta = \mu * \psi_j \in \mathrm{P}(G)$, we have

$$\sup_{g\in Q} \|\lambda(g)\zeta - \zeta\|_1 \leq \varepsilon^2.$$

Set $\xi = \zeta^{1/2} \in \mathrm{L}^2(G)_+$ and observe that $\|\xi\| = 1$. Moreover, we have

$$\begin{aligned}
\sup_{g\in Q} \|\lambda_G(g)\xi - \xi\|_2^2 &= \sup_{g\in Q} \int_G |\xi(g^{-1}h) - \xi(h)|^2 \, \mathrm{d}m_G(h) \\
&= \sup_{g\in Q} \int_G |\zeta(g^{-1}h)^{1/2} - \zeta(h)^{1/2}|^2 \, \mathrm{d}m_G(h) \\
&\leq \sup_{g\in Q} \int_G |\zeta(g^{-1}h) - \zeta(h)| \, \mathrm{d}m_G(h) \\
&= \sup_{g\in Q} \|\lambda(g)\zeta - \zeta\|_1 \leq \varepsilon^2.
\end{aligned}$$

This implies that $1_G \prec \lambda_G$ and finishes the proof of Theorem 2.20. $\square$

For countable discrete groups, we prove the following dynamical characterization of amenability.

Theorem 2.23. *Let Γ be any countable discrete group. The following assertions are equivalent:*

(i) *Γ is amenable.*
(ii) *For any action $\Gamma \curvearrowright X$ by homeomorphisms on a compact metrizable space, there exists a Γ-invariant Borel probability measure $\nu \in \mathrm{Prob}(X)$.*

Proof. (i) $\Rightarrow$ (ii) Denote by $\mathrm{Prob}(X) \subset \mathrm{C}(X)^*$ the convex weak*-compact subset of all Borel probability measures on X and consider the affine action $\Gamma \curvearrowright \mathrm{Prob}(X)$. Since Γ is amenable, there exists a Γ-invariant Borel probability measure $\nu \in \mathrm{Prob}(X)$.

(ii) $\Rightarrow$ (i) Denote by $\mathcal{M}$ the convex weak*-compact set of all means on $\ell^\infty(\Gamma)$. As $\ell^\infty(\Gamma)$ is not $\|\cdot\|_\infty$-separable, we cannot directly use the assumption in item (ii). However, since Γ is countable, there exists an increasing net $(\mathcal{F}_i)_{i\in I}$ of Γ-invariant closed $\|\cdot\|_\infty$-separable subspaces of $\ell^\infty(\Gamma)$ so that $\mathbb{C}\mathbf{1}_\Gamma \subset \mathcal{F}_i$ for every $i \in I$ and $\ell^\infty(\Gamma) = \overline{\bigcup_{i\in I} \mathcal{F}_i}$. For every $i \in I$, denote by $\mathcal{M}_i \subset (\mathcal{F}_i)^*$ the weak*-compact convex subset of all means on $\mathcal{F}_i$ and by $r_i : \mathcal{M} \to \mathcal{M}_i : \mathfrak{m} \mapsto \mathfrak{m}|_{\mathcal{F}_i}$ the Γ-equivariant restriction map. Then we have $\mathcal{M}^\Gamma = \bigcap_{i\in I} r_i^{-1}(\mathcal{M}_i^\Gamma)$. Note that $\mathcal{M}_i$ is metrizable since $\mathcal{F}_i$ is $\|\cdot\|_\infty$-separable. By assumption, there exists a Γ-invariant Borel probability measure $\mu_i \in \mathrm{Prob}(\mathcal{M}_i)$. Arguing as in the proof of Theorem 2.20 (iii) $\Rightarrow$ (iv), we see that $\mathrm{Bar}(\mu_i) \in \mathcal{M}_i^\Gamma$ is a left Γ-invariant mean.

We have showed that for every $i \in I$, the compact subset $\mathcal{M}_i^\Gamma \subset \mathcal{M}$ is nonempty. Moreover, for any finite subset $F \subset I$, since I is a directed set, there exists $j \in I$ such that $i \leq j$ for every $i \in F$. This implies that $\mathcal{M}_j^\Gamma \subset \bigcap_{i \in F} \mathcal{M}_i^\Gamma$ and so $\bigcap_{i \in F} r_i^{-1}(\mathcal{M}_i^\Gamma)$ is nonempty. Since $\mathcal{M}$ is compact, it follows that $\mathcal{M}^\Gamma = \bigcap_{i \in I} r_i^{-1}(\mathcal{M}_i^\Gamma)$ is nonempty. This shows that $\ell^\infty(\Gamma)$ has a left invariant mean and thus Γ is amenable by Theorem 2.20. $\square$

We conclude this section by proving von Neumann's result regarding nonamenability of free groups.

THEOREM 2.24 (von Neumann). *Denote by $\mathbf{F}_2 = \langle a, b \rangle$ the free group on two generators. Then $\mathbf{F}_2$ is nonamenable.*

PROOF. By contradiction, assume that $\mathbf{F}_2 = \langle a, b \rangle$ is amenable. Denote by $\mathfrak{m} \in \ell^\infty(\mathbf{F}_2)^*$ a left invariant mean. Define $\mathfrak{n} : \mathcal{P}(\mathbf{F}_2) \to [0,1] : W \mapsto \mathfrak{m}(\mathbf{1}_W)$ and observe that $\mathfrak{n}$ is a finitely additive left invariant *probability mean* on $\mathbf{F}_2$. Then we necessarily have $\mathfrak{n}(F) = 0$ for every finite subset $F \subset \mathbf{F}_2$. In particular, we have $\mathfrak{n}(\{e\}) = 0$.

Denote by $W_a \subset \mathbf{F}_2$ the subset of reduced words whose first letter is a. Likewise, consider the subsets $W_{a^{-1}}, W_b, W_{b^{-1}} \subset \mathbf{F}_2$. Observe that $\mathbf{F}_2 \setminus \{e\} = W_a \sqcup W_{a^{-1}} \sqcup W_b \sqcup W_{b^{-1}}$. Since $a \cdot (W_a \sqcup W_b \sqcup W_{b^{-1}}) \subset W_a$, it follows that

$$\begin{aligned}\mathfrak{n}(W_a) + \mathfrak{n}(W_b) + \mathfrak{n}(W_{b^{-1}}) &= \mathfrak{n}(W_a \sqcup W_b \sqcup W_{b^{-1}}) \\ &= \mathfrak{n}(a \cdot (W_a \sqcup W_b \sqcup W_{b^{-1}})) \\ &\leq \mathfrak{n}(W_a).\end{aligned}$$

This implies that $\mathfrak{n}(W_b) = \mathfrak{n}(W_{b^{-1}}) = 0$. Likewise, we have $\mathfrak{n}(W_a) = \mathfrak{n}(W_{a^{-1}}) = 0$. This further implies that $\mathfrak{n}(\mathbf{F}_2) = 0$, a contradiction. $\square$

One can show that amenability is inherited by closed subgroups. Thus, any locally compact group that contains $\mathbf{F}_2$ as a closed subgroup is nonamenable.

3. Property (T)

DEFINITION 2.25 (Kazhdan [**Ka67**]). Let G be any locally compact group. We say that G has *property* (T) if for every strongly continuous unitary representation $\pi : G \to \mathcal{U}(\mathcal{H}_\pi)$ such that $1_G \prec \pi$, we have $1_G \subset \pi$.

First, we prove various permanence properties enjoyed by locally compact groups with property (T).

PROPOSITION 2.26. *Let G, H be any locally compact groups. Assume that G has property* (T)*. The following assertions hold:*

(i) *If $\rho : G \to H$ is a continuous homomorphism with dense range, then H has property* (T)*.*

(ii) *If* $H \lhd G$ *is a closed normal subgroup , then* G/H *has property* (T).
(iii) G *is compactly generated. In particular, if* G *is discrete, then* G *is finitely generated.*
(iv) G *is unimodular.*

PROOF. (i) Let $\pi : H \to \mathcal{U}(\mathcal{H}_\pi)$ be any strongly continuous unitary representation such that $1_H \prec \pi$. Then $\pi \circ \rho : G \to \mathcal{U}(\mathcal{H}_\pi)$ is a strongly continuous unitary representation such that $1_G \prec \pi \circ \rho$. Since G has property (T), we have $1_G \subset \pi \circ \rho$ and so π has a nonzero $\rho(G)$-invariant vector. By continuity and density of $\rho(G)$ in H, it follows that $1_H \subset \pi$. This shows that H has property (T).

(ii) It suffices to apply item (i) to the continuous homomorphism $G \to G/H$.

(iii) Denote by $\mathcal{O}$ the set of all compactly generated open subgroups of G. Since G is locally compact, we have $\mathcal{O} \neq \emptyset$ and $G = \bigcup_{H \in \mathcal{O}} H$. For every $H \in \mathcal{O}$, since H is open in G, the homogeneous space G/H is discrete. Denote by $\pi : G \to \mathcal{U}(\mathcal{H}_\pi)$ the strongly continuous unitary representation where $\mathcal{H}_\pi \doteq \bigoplus_{H \in \mathcal{O}} \ell^2(G/H)$ and such that

$$\forall g, h \in G, \forall H \in \mathcal{O}, \quad \pi(g)\delta_{hH} = \delta_{ghH}.$$

We claim that $1_G \prec \pi$. Indeed, let $Q \subset G$ be any compact subset. By compactness, there exist $H_1, \dots, H_k \in \mathcal{O}$ such that $Q \subset H_1 \cup \cdots \cup H_k$. Denote by $H < G$ the subgroup generated by $H_1, \dots, H_k$ and observe that $H \in \mathcal{O}$. For every $g \in Q$, since $Q \subset H$, we have $\pi(g)\delta_H = \delta_H$. This shows that $1_G \prec \pi$. Since G has property (T), there exists a nonzero $\pi(G)$-invariant vector $\xi \in \mathcal{H}_\pi$. Then there exists $H \in \mathcal{O}$ such that the orthogonal projection $\xi_H \in \ell^2(G/H)$ of $\xi \in \mathcal{H}_\pi$ is nonzero. Since $\pi(g)\xi_H = \xi_H$ for every $g \in G$ and since $\xi_H \neq 0$, it follows that G/H is finite. Since $H < G$ is compactly generated, it follows that G is compactly generated.

(iv) Denote by $\Delta_G : G \to \mathbb{R}^*_+$ the modular function. Then $\overline{\Delta_G(G)}$ has property (T) by (iii). Since $\overline{\Delta_G(G)}$ is abelian, $\overline{\Delta_G(G)}$ is amenable by Proposition 2.17 and so $\overline{\Delta_G(G)}$ is compact (see Proposition 2.27 below). It follows that $\Delta_G(G) = \{1\}$ and thus G is unimodular. □

Next, we observe that property (T) is completely opposite to amenability. In particular, we obtain the following characterization of compact groups.

PROPOSITION 2.27. *Let* G *be any locally compact group. The following assertions are equivalent:*

(i) G *is compact.*
(ii) G *is amenable and has property* (T).

PROOF. (i) $\Rightarrow$ (ii) Assume that G is compact. Then G is amenable by Proposition 2.16. Let now $\pi : G \to \mathcal{U}(\mathcal{H}_\pi)$ be any strongly continuous unitary representation such that $1_G \prec \pi$. There exists a unit vector $\xi \in \mathcal{H}_\pi$ such that $\sup_{g \in G} \|\pi(g)\xi - \xi\| \leq 1/2$. Denote by $\eta \in \mathcal{H}_\pi$ the circumcenter of the bounded set $\pi(G)\xi$. Then η is $\pi(G)$-invariant. Moreover, since η belongs to the closure of the convex hull of $\pi(G)\xi$, it follows that $\|\eta - \xi\| \leq 1/2$ and so $\eta \neq 0$. This shows that $1_G \subset \pi$.

(ii) $\Rightarrow$ (i) Since G has property (T), G is compactly generated by Proposition 2.26(iii) and hence σ-compact. Since G is amenable, we have $1_G \prec \lambda_G$ by Theorem 2.20. Since G has property (T), we obtain $1_G \subset \lambda_G$. Proposition 2.4 implies that G is compact. ☐

Next we show that property (T) is inherited by lattices.

PROPOSITION 2.28. *Let G be any locally compact second countable group and $\Gamma < G$ any lattice. If G has property* (T)*, then so does Γ.*

PROOF. Let $\pi : \Gamma \to \mathcal{U}(\mathcal{H}_\pi)$ be any unitary representation such that $1_\Gamma \prec \pi$. Denote by $\widehat{\pi} : G \to \mathcal{U}(\mathcal{H}_{\widehat{\pi}})$ the induced representation. We choose the first viewpoint on induction. Set $X = G/\Gamma$. We may choose a Borel section $\sigma : X \to G$ as in Corollary 1.12 such that $\sigma(K)$ is relatively compact in G for every compact subset $K \subset X$. As usual, denote by $\tau : G \times X \to \Gamma : (g, x) \mapsto \sigma(gx)^{-1} g\sigma(x)$ the corresponding Borel 1-cocycle.

CLAIM 2.29. We have $1_G \prec \widehat{\pi}$.

Indeed, let $Q \subset G$ be any compact subset and $\varepsilon > 0$. We may assume that $e \in Q$. Choose a compact subset $K \subset X$ such that $\nu(X \setminus K) < \frac{\varepsilon^2}{8}$. Since the action map $G \times X \to X : (g, x) \mapsto gx$ is continuous, the subset $Q^{-1}K$ is compact in X. This implies that the image of the map $f : Q \times X \to G : (g, x) \mapsto \tau(g, g^{-1}x)$ is relatively compact in G. Since Γ is discrete in G, this further implies that $\Lambda \doteq f(Q \times K) \cap \Gamma$ is a finite subset of Γ. Since $1_\Gamma \prec \pi$, there exists a unit vector $\xi \in \mathcal{H}_\pi$ such that

$$\max\{\|\pi(\gamma)\xi - \xi\| \mid \gamma \in \Lambda\} < \frac{\varepsilon}{\sqrt{2}}.$$

Set $\eta = \mathbf{1}_X \otimes \xi \in \mathcal{H}_{\widehat{\pi}}$. Then $\|\eta\| = 1$ and for every $g \in Q$, we have

$$\begin{aligned}
\|\widehat{\pi}(g)\eta - \eta\|^2 &= \int_X \|\pi(\tau(g, g^{-1}x))\xi - \xi\|^2 \, \mathrm{d}\nu(x) \\
&\leq \int_K \|\pi(\tau(g, g^{-1}x))\xi - \xi\|^2 \, \mathrm{d}\nu(x) + 4\nu(X \setminus K) \\
&\leq \max\{\|\pi(\gamma)\xi - \xi\|^2 \mid \gamma \in \Lambda\} + 4\nu(X \setminus K) \\
&< \varepsilon^2.
\end{aligned}$$

This shows that $1_G \prec \widehat{\pi}$ and finishes the proof of Claim 2.29.

Since G has property (T), we obtain $1_G \subset \widehat{\pi}$. Proposition 2.13 further implies that $1_\Gamma \subset \pi$. ☐

We point out that the converse to Proposition 2.28 holds, namely if $\Gamma < G$ is a lattice and if Γ has property (T), then G has property (T). We will not prove this fact.

Corollary 2.30. *Let G be any locally compact second countable group with property* (T). *Then any lattice $\Gamma < G$ is weakly uniform.*

Proof. The strongly continuous unitary representation $\lambda^0_{G/\Gamma} : G \to \mathcal{U}(\mathrm{L}^2(G/\Gamma)^0)$ is ergodic by Proposition 2.7. By property (T), $\lambda^0_{G/\Gamma}$ has spectral gap. This means that $\Gamma < G$ is weakly uniform. $\square$

4. Property (T) for $\mathrm{SL}_d(\mathbb{R})$, $d \geq 3$

4.1. Howe–Moore property for $\mathrm{SL}_d(\mathbb{R})$, $d \geq 2$. Let $\mathcal{H}$ be any (complex) Hilbert space and denote by $\mathbb{B}(\mathcal{H})$ the unital Banach $*$-algebra of all bounded linear operators $T : \mathcal{H} \to \mathcal{H}$. Besides the norm topology on $\mathbb{B}(\mathcal{H})$ given by the supremum norm

$$\forall T \in \mathbb{B}(\mathcal{H}), \quad \|T\|_\infty \doteq \sup \{\|T\xi\| \mid \xi \in \mathcal{H}, \|\xi\| \leq 1\},$$

we can define two weaker locally convex topologies on $\mathbb{B}(\mathcal{H})$ as follows.

- The *strong operator topology* on $\mathbb{B}(\mathcal{H})$ is defined as the initial topology on $\mathbb{B}(\mathcal{H})$ that makes the maps $\mathbb{B}(\mathcal{H}) \to \mathbb{C} : T \mapsto \|T\xi\|$ continuous for all $\xi \in \mathcal{H}$.
- The *weak operator topology* on $\mathbb{B}(\mathcal{H})$ is defined as the initial topology on $\mathbb{B}(\mathcal{H})$ that makes the maps $\mathbb{B}(\mathcal{H}) \to \mathbb{C} : T \mapsto |\langle T\xi, \eta\rangle|$ continuous for all $\xi, \eta \in \mathcal{H}$.

Note that we already defined the strong operator topology on $\mathcal{U}(\mathcal{H})$. As a matter of fact, on $\mathcal{U}(\mathcal{H})$, strong and weak operator topologies coincide. Observe that when $\mathcal{H}$ is separable, both strong and weak operator topologies are metrizable on the unit ball of $\mathbb{B}(\mathcal{H})$ denoted by $\mathrm{Ball}(\mathbb{B}(\mathcal{H}))$. Moreover, $\mathrm{Ball}(\mathbb{B}(\mathcal{H}))$ is weakly compact.

Let G be any locally compact group and $\pi : G \to \mathcal{U}(\mathcal{H}_\pi)$ any strongly continuous unitary representation. We say that π is *mixing* if $\pi(g) \to 0$ weakly as $g \to \infty$. Note that when G is noncompact, the left regular representation $\lambda_G : G \to \mathcal{U}(\mathrm{L}^2(G))$ is mixing. Any mixing strongly continuous unitary representation is ergodic. In that respect, we introduce the following terminology.

Definition 2.31. Let G be any noncompact locally compact group. We say that G has the *Howe–Moore property* if any ergodic strongly continuous unitary representation $\pi : G \to \mathcal{U}(\mathcal{H}_\pi)$ is mixing.

Observe that when G has the Howe–Moore property, for every nontrivial strongly continuous unitary representation $\pi : G \to \mathcal{U}(\mathcal{H}_\pi)$, the subrepresentation $\pi^0 : G \to \mathcal{U}(\mathcal{H}_\pi \ominus (\mathcal{H}_\pi)^G)$ is ergodic and hence mixing. Here are some properties enjoyed by locally compact groups with the Howe–Moore property.

Proposition 2.32. *Let G be any noncompact locally compact group with the Howe–Moore property. The following assertions hold:*

(i) *For every closed normal subgroup $N \lhd G$, either N is compact or $N = G$.*
(ii) *For every open subgroup $H < G$, either H is compact or $H = G$.*
(iii) *For every ergodic pmp action $G \curvearrowright (X, \nu)$ and every noncompact closed subgroup $H < G$, the action $H \curvearrowright (X, \nu)$ is still ergodic.*

Proof. (i) Let $N \lhd G$ be any proper closed normal subgroup. Define the quasi-regular representation $\pi : G \to \mathcal{U}(\mathrm{L}^2(G/N))$ and note that $\pi = \lambda_{G/N} \circ p$ where $p : G \to G/N$ is the canonical factor map and $\lambda_{G/N} : G/N \to \mathcal{U}(\mathrm{L}^2(G/N))$ is the left regular representation of the locally compact group G/N. Since $N \neq G$, we have $\mathrm{L}^2(G/N)^G \neq \mathrm{L}^2(G/N)$. By Howe–Moore property, the subrepresentation $\pi^0 : G \to \mathcal{U}(\mathrm{L}^2(G/N) \ominus \mathrm{L}^2(G/N)^G)$ is mixing. Since $\pi|_N \equiv 1$, it follows that $\pi^0|_N \equiv 1$ and thus N is compact.

(ii) Let $H < G$ be any proper open subgroup. Then the homogeneous space G/H is discrete and nontrivial. Define the strongly continuous unitary representation $\pi : G \to \mathcal{U}(\ell^2(G/H))$ by the formula

$$\forall g, h \in G, \quad \pi(g)\delta_{hH} = \delta_{ghH}.$$

Since $H \neq G$, the unit vector $\delta_H \in \ell^2(G/H)$ is not $\pi(G)$-invariant and so $\ell^2(G/H)^G \neq \ell^2(G/H)$. By Howe–Moore property, the subrepresentation $\pi^0 : G \to \mathcal{U}(\ell^2(G/H) \ominus \ell^2(G/H)^G)$ is mixing. Since the nonzero vector $\xi \doteq \delta_H - P_{\ell^2(G/H)^G}(\delta_H) \in \ell^2(G/H) \ominus \ell^2(G/H)^G$ is $\pi(H)$-invariant, it follows that H is compact.

(iii) Let $H < G$ be any noncompact closed subgroup and $G \curvearrowright (X, \nu)$ any ergodic pmp action. By Proposition 2.7, the Koopman representation $\kappa^0 : G \to \mathcal{U}(\mathrm{L}^2(X, \nu)^0)$ is ergodic. By Howe–Moore property, $\kappa^0 : G \to \mathcal{U}(\mathrm{L}^2(X, \nu)^0)$ is mixing and hence $\pi|_H : H \to \mathcal{U}(\mathrm{L}^2(X, \nu)^0)$ is ergodic. Then Proposition 2.7 implies that $H \curvearrowright (X, \nu)$ is ergodic. $\square$

The main theorem of this subsection is the following well-known result due to Howe–Moore.

Theorem 2.33 (Howe–Moore [**HM77**]). *For every $d \geq 2$, $\mathrm{SL}_d(\mathbb{R})$ has the Howe–Moore property.*

As a consequence of Theorem 2.33 and Proposition 2.32(iii), we obtain the following ergodicity result due to Moore.

Corollary 2.34 (Moore [**Mo65**]). *Let $d \geq 2$ and set $G = \mathrm{SL}_d(\mathbb{R})$. Let $\Gamma < G$ be any lattice and denote by $\nu \in \mathrm{Prob}(G/\Gamma)$ the unique G-invariant Borel probability measure. For every noncompact closed subgroup $H < G$, the pmp action $H \curvearrowright (G/\Gamma, \nu)$ is ergodic.*

Before proving Theorem 2.33, we need to prove some preliminary results that are also of independent interest.

Define the following subgroups of $\mathrm{SL}_2(\mathbb{R})$:

$$U^+ = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in \mathbb{R} \right\}$$

$$U^- = \left\{ \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \mid x \in \mathbb{R} \right\}$$
$$A = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \mid \lambda > 0 \right\}.$$

Observe that $\mathrm{SL}_2(\mathbb{R})$ is generated by $U^+ \cup U^-$.

LEMMA 2.35. *Let $\pi : \mathrm{SL}_2(\mathbb{R}) \to \mathcal{U}(\mathcal{H}_\pi)$ be any strongly continuous unitary representation. Every $\pi(U^+)$-invariant vector is $\pi(\mathrm{SL}_2(\mathbb{R}))$-invariant.*

PROOF. Let $\xi \in \mathcal{H}_\pi$ be any $\pi(U^+)$-invariant unit vector. Define the continuous function $\varphi : G \to \mathbb{C} : g \mapsto \langle \pi(g)\xi, \xi \rangle$. By assumption, φ is U^+-bi-invariant. For every $n \geq 1$, set

$$g_n \doteq \begin{pmatrix} 0 & -n \\ \frac{1}{n} & 0 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}).$$

A simple calculation shows that for every $\lambda > 0$, we have

$$\begin{pmatrix} 1 & \lambda n \\ 0 & 1 \end{pmatrix} g_n \begin{pmatrix} 1 & \frac{n}{\lambda} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ \frac{1}{n} & \lambda^{-1} \end{pmatrix} \to \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}.$$

Since φ is continuous and U^+-bi-invariant, it follows that

$$\forall a \in A, \quad \varphi(a) = \lim_n \varphi(g_n) = \varphi(1) = 1.$$

This further implies that $\pi(a)\xi = \xi$ for every $a \in A$. It follows that φ is A-bi-invariant.

Another simple calculation shows that for every $x \in \mathbb{R}$, we have

$$\begin{pmatrix} n & 0 \\ 0 & \frac{1}{n} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{n} & 0 \\ 0 & n \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{x}{n^2} & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Since φ is continuous and A-bi-invariant, it follows that for every $u \in U^-$, we have $\varphi(u) = 1$ and so $\pi(u)\xi = \xi$.

We have showed that ξ is both $\pi(U^+)$-invariant and $\pi(U^-)$-invariant. Since $\mathrm{SL}_2(\mathbb{R})$ is generated by $U^+ \cup U^-$, it follows that ξ is $\pi(\mathrm{SL}_2(\mathbb{R}))$-invariant. $\square$

Let $d \geq 2$. For all $1 \leq a \neq b \leq d$ and all $x \in \mathbb{R}$, denote by $E_{ab}(x) \in \mathrm{SL}_d(\mathbb{R})$ the elementary matrix defined by $(E_{ab}(x))_{ij} = 1$ if $i = j$, $(E_{ab}(x))_{ij} = x$ if $i = a$ and $j = b$, $(E_{ab}(x))_{ij} = 0$ otherwise. We leave as an exercise to check that $\mathrm{SL}_d(\mathbb{R})$ is generated by $\{E_{ab}(x) \mid 1 \leq a \neq b \leq d, x \in \mathbb{R}\}$. For every $2 \leq k \leq d$, regard $\mathrm{SL}_k(\mathbb{R}) < \mathrm{SL}_d(\mathbb{R})$ as the following subgroup:

$$\mathrm{SL}_k(\mathbb{R}) \cong \left\{ \begin{pmatrix} A & 0_{d-k,k} \\ 0_{k,d-k} & 1_{d-k,d-k} \end{pmatrix} \mid A \in \mathrm{SL}_k(\mathbb{R}) \right\}.$$

For all $1 \leq \ell_1 < \ell_2 \leq d$, denote by $H_{\ell_1,\ell_2} < \mathrm{SL}_d(\mathbb{R})$ the (ℓ_1, ℓ_2)-*copy* of $\mathrm{SL}_2(\mathbb{R})$ in $\mathrm{SL}_d(\mathbb{R})$ that consists in all matrices $g \in \mathrm{SL}_d(\mathbb{R})$ such that $g_{\ell_1\ell_1} =$

α, $g_{\ell_1\ell_2} = \beta$, $g_{\ell_2\ell_1} = \gamma$, $g_{\ell_2\ell_2} = \delta$, $g_{ii} = 1$ for all $i \neq \ell_1, \ell_2$, $g_{ij} = 0$ for all $i \neq j$ and $\{i,j\} \neq \{\ell_1, \ell_2\}$ and such that

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}).$$

LEMMA 2.36. *Let $d \geq 2$ and $\pi : \mathrm{SL}_d(\mathbb{R}) \to \mathcal{U}(\mathcal{H}_\pi)$ be any strongly continuous unitary representation. Let $\xi \in \mathcal{H}_\pi$ be any $\pi(H_{\ell_1,\ell_2})$-invariant vector for some $1 \leq \ell_1 < \ell_2 \leq d$. Then ξ is $\pi(\mathrm{SL}_d(\mathbb{R}))$-invariant.*

PROOF. Up to permutation, we may assume that $\ell_1 = 1$ and $\ell_2 = 2$. We proceed by induction over $2 \leq k \leq d$. By assumption, ξ is $\pi(\mathrm{SL}_2(\mathbb{R}))$-invariant. Assume that ξ is $\pi(\mathrm{SL}_k(\mathbb{R}))$-invariant for $2 \leq k \leq d-1$ and let us show that ξ is $\pi(\mathrm{SL}_{k+1}(\mathbb{R}))$-invariant. Let $1 \leq j \leq k$ and $x \in \mathbb{R}$. For every $n \geq 1$, denote by $g_n \in \mathrm{SL}_k(\mathbb{R}) < \mathrm{SL}_{k+1}(\mathbb{R})$ any diagonal matrix such that $(g_n)_{ii} = \frac{1}{n}$ if $i = j$. Then a simple computation shows that $g_n E_{j(k+1)}(x) g_n^{-1} = E_{j(k+1)}(\frac{x}{n}) \to 1$ as $n \to \infty$ and $g_n^{-1} E_{(k+1)j}(x) g_n = E_{(k+1)j}(\frac{x}{n}) \to 1$ as $n \to \infty$. Since $\pi(g_n)\xi = \xi$, we have

$$\begin{aligned} \|\pi(E_{j(k+1)}(x))\xi - \xi\| &= \lim_n \|\pi(E_{j(k+1)}(x))\pi(g_n)^*\xi - \pi(g_n)^*\xi\| \\ &= \lim_n \|\pi(g_n E_{j(k+1)}(x) g_n^{-1})\xi - \xi\| = 0 \end{aligned}$$

and so $\pi(E_{j(k+1)}(x))\xi = \xi$. Likewise, we have $\pi(E_{(k+1)j}(x))\xi = \xi$. Since $\mathrm{SL}_{k+1}(\mathbb{R})$ is generated by $\mathrm{SL}_k(\mathbb{R}) \cup \{E_{j(k+1)}(x), E_{(k+1)j}(x) \mid 1 \leq j \leq k, x \in \mathbb{R}\}$, it follows that ξ is $\pi(\mathrm{SL}_{k+1}(\mathbb{R}))$-invariant. By induction over $2 \leq k \leq d$, it follows that ξ is $\pi(\mathrm{SL}_d(\mathbb{R}))$-invariant. $\square$

Let $d \geq 2$. Denote by $K \doteq \mathrm{SO}_d(\mathbb{R}) < \mathrm{SL}_d(\mathbb{R})$ the special orthogonal subgroup and observe that $K < \mathrm{SL}_d(\mathbb{R})$ is compact. Define the subset $A^+ \subset \mathrm{SL}_d(\mathbb{R})$ of diagonal matrices by

$$A^+ \doteq \{\mathrm{diag}(\lambda_1, \dots, \lambda_d) \mid \lambda_1 \geq \cdots \geq \lambda_d > 0,\ \lambda_1 \cdots \lambda_d = 1\} \subset \mathrm{SL}_d(\mathbb{R})$$

and by $A < \mathrm{SL}_d(\mathbb{R})$ the subgroup of diagonal matrices generated by A^+.

LEMMA 2.37 (Cartan decomposition). *We have $\mathrm{SL}_d(\mathbb{R}) = K \cdot A^+ \cdot K$.*

PROOF. Let $g \in \mathrm{SL}_d(\mathbb{R})$ be any matrix. By polar decomposition, we may write $g = k_0 h$ where $k_0 \in K$ and $h \in \mathrm{SL}_d(\mathbb{R})$ is symmetric positive definite. By diagonalization, there exists $k_2 \in K$ such that $k_2 h k_2^{-1} = a \in A^+$. Then $g = k_1 a k_2$ with $k_1 = k_0 k_2^{-1} \in K$. $\square$

We now have all the tools to prove Theorem 2.33.

PROOF OF THEOREM 2.33. Let $d \geq 2$ and $\pi : \mathrm{SL}_d(\mathbb{R}) \to \mathcal{U}(\mathcal{H}_\pi)$ be any strongly continuous unitary representation. Assuming that π is not mixing, we show that there exists a nonzero $\pi(\mathrm{SL}_d(\mathbb{R}))$-invariant vector. Since $\mathrm{SL}_d(\mathbb{R})$ is second countable, $\pi(G)\xi$ is separable for every $\xi \in \mathcal{H}_\pi$ and so we may assume assume that $\mathcal{H}_\pi$ is separable. Since π is not mixing, there exists a sequence $(g_n)_{n\in\mathbb{N}}$ in G such that $g_n \to \infty$ and $\pi(g_n) \not\to 0$ weakly. Up to taking a subsequence, we may assume that there exists $T \in \mathbb{B}(\mathcal{H})$ such that $T \neq 0$ and $\pi(g_n) \to T$ weakly. Using Lemma 2.37, there exist sequences $(k_{1,n})_{n\in\mathbb{N}}$ and $(k_{2,n})_{n\in\mathbb{N}}$ in K and $(a_n)_{n\in\mathbb{N}}$ in A^+ such that $g_n = k_{1,n}a_nk_{2,n}$ for every $n \in \mathbb{N}$. Up to taking another subsequence, we may assume that $k_{1,n} \to k_1$ in K and $k_{2,n} \to k_2$ in K. This implies that $\pi(k_{1,n}) \to \pi(k_1)$ and $\pi(k_{2,n}) \to \pi(k_2)$ strongly. This further implies that $\pi(a_n) \to \pi(k_1)^*T\pi(k_2)^*$ weakly. Set $S \doteq \pi(k_1)^*T\pi(k_2)^* \in \mathbb{B}(\mathcal{H})$ and observe that $S \neq 0$.

For every $n \in \mathbb{N}$, write $a_n = \mathrm{diag}(\lambda_{1,n}, \dots, \lambda_{d,n})$ with $\lambda_{1,n} \geq \cdots \geq \lambda_{d,n}$ and $\lambda_{1,n} \cdots \lambda_{d,n} = 1$. Since $a_n \to \infty$, it follows that $\frac{\lambda_{1,n}}{\lambda_{d,n}} \to +\infty$. A simple computation shows that for every $x \in \mathbb{R}$,

$$a_n^{-1}E_{1d}(x)a_n = E_{1d}(\frac{\lambda_{d,n}}{\lambda_{1,n}}x) \to 1.$$

This implies that for every $x \in \mathbb{R}$, we have $\pi(E_{1d}(x))S = S$ since

$$\begin{aligned}\forall \eta_1, \eta_2 \in \mathcal{H}_\pi, \quad \langle \pi(E_{1d}(x))S\eta_1, \eta_2\rangle &= \lim_n \langle \pi(E_{1d}(x))\pi(a_n)\eta_1, \eta_2\rangle \\ &= \lim_n \langle \pi(a_n^{-1}E_{1d}(x)a_n)\eta_1, \pi(a_n^{-1})\eta_2\rangle \\ &= \langle \eta_1, S^*\eta_2\rangle \\ &= \langle S\eta_1, \eta_2\rangle.\end{aligned}$$

Choose $\eta \in \mathcal{H}_\pi$ so that $\xi \doteq S\eta \neq 0$. Then $\xi \in \mathcal{H}_\pi$ is a nonzero $\pi(E_{1d}(\mathbb{R}))$-invariant vector. Denote by $H_{1d} < \mathrm{SL}_d(\mathbb{R})$ the $(1, d)$-copy of $\mathrm{SL}_2(\mathbb{R})$. By Lemma 2.35, ξ is $\pi(H_{1d})$-invariant and by Lemma 2.36, ξ is $\pi(\mathrm{SL}_d(\mathbb{R}))$-invariant. This finishes the proof of Theorem 2.33. $\square$

4.2. Property (T) for $\mathrm{SL}_d(\mathbb{R})$, $d \geq 3$. The main theorem of this subsection is the following celebrated result due to Kazhdan.

THEOREM 2.38 (Kazhdan [**Ka67**]). *For every $d \geq 3$, $\mathrm{SL}_d(\mathbb{R})$ has property* (T).

Before proving Theorem 2.38, we need to prove some preliminary results that are also of independent interest.

For any locally compact second countable abelian group N, we denote by $\widehat{N}$ the *unitary dual* of N

$$\widehat{N} = \{\chi : N \to \mathbb{T} \mid \chi \text{ is a continuous group homomorphism}\}.$$

Endowed with the topology of uniform convergence on compact subsets, $\widehat{N}$ is a locally compact second countable abelian group. We refer to [**BHV08**, Section A.2] for further details. Denote by $\mathcal{B}(\widehat{N})$ the σ-algebra of Borel subsets of $\widehat{N}$. For every regular Borel probability measure $\mu \in \mathrm{Prob}(\widehat{N})$, define the strongly continuous unitary representation $\pi_\mu : N \to \mathcal{U}(\mathrm{L}^2(\widehat{N}, \mu))$ by the formula

$$\forall \xi \in \mathrm{L}^2(\widehat{N}, \mu), \forall g \in N, \quad (\pi_\mu(g)\xi)(\chi) = \chi(g)\,\xi(\chi).$$

The fact that $\pi_\mu : N \to \mathcal{U}(\mathrm{L}^2(\widehat{N}, \mu))$ is strongly continuous follows from Lemma 2.2 after noticing that for all $\xi_1, \xi_2 \in \mathrm{L}^2(\widehat{N}, \mu)$, the map

$$\varphi_{\xi_1,\xi_2} : N \to \mathbb{C} : g \mapsto \int_{\widehat{N}} \chi(g)\xi_1(\chi)\,\overline{\xi_2(\chi)}\,\mathrm{d}\mu(\chi)$$

is continuous (hence measurable) thanks to Lebesgue's dominated convergence theorem.

We prove the following spectral theorem regarding strongly continuous unitary representations of locally compact abelian groups.

THEOREM 2.39. *Let N be any locally compact second countable abelian group and $\pi : N \to \mathcal{U}(\mathcal{H}_\pi)$ be any strongly continuous unitary representation. Then there exists a unique mapping $E_\pi : \mathcal{B}(\widehat{N}) \to \mathbb{B}(\mathcal{H}_\pi)$ that satisfies the following properties.*

(i) *For every $\xi \in \mathcal{H}_\pi$, the mapping $\mathcal{B}(\widehat{N}) \to \mathbb{R}_+ : B \mapsto \langle E_\pi(B)\xi, \xi\rangle$ defines a finite regular Borel measure μ_ξ on $\widehat{N}$ such that*

$$\forall h \in N, \quad \langle \pi(h)\xi, \xi\rangle = \int_{\widehat{N}} \chi(h)\,\mathrm{d}\mu_\xi(\chi).$$

(ii) *For every $B \in \mathcal{B}(\widehat{N})$, $E_\pi(B)$ is an orthogonal projection in $\mathbb{B}(\mathcal{H}_\pi)$. Moreover, $E_\pi(\{1_{\widehat{N}}\})$ is the orthogonal projection onto the closed subspace $(\mathcal{H}_\pi)^N$ of $\pi(N)$-invariant vectors.*

We then say that $E_\pi : \mathcal{B}(\widehat{N}) \to \mathbb{B}(\mathcal{H}_\pi)$ is the projection-valued *spectral measure* associated with $\pi : N \to \mathcal{U}(\mathcal{H}_\pi)$.

PROOF. (i) Using Bochner's theorem (see [**BHV08**, Theorem D.2.2]), for every $\xi \in \mathcal{H}_\pi$, there exists a finite regular Borel measure μ_ξ on $\widehat{N}$ such that

$$\forall h \in N, \quad \langle \pi(h)\xi, \xi\rangle = \int_{\widehat{N}} \chi(h)\,\mathrm{d}\mu_\xi(\chi).$$

For all $\xi, \eta \in \mathcal{H}_\pi$, define the finite regular complex Borel measure $\mu_{\xi,\eta}$ on $\widehat{N}$ by the formula $\mu_{\xi,\eta} = \frac{1}{4}\sum_{k=0}^{3} \mathrm{i}^k \mu_{\xi+\mathrm{i}^k\eta}$. Then we have

$$\forall h \in N, \quad \langle \pi(h)\xi, \eta\rangle = \int_{\widehat{H}} \chi(h)\,\mathrm{d}\mu_{\xi,\eta}(\chi).$$

Observe that for every $\xi \in \mathcal{H}_\pi$ with $\|\xi\| = 1$, we have $\mu_\xi \in \mathrm{Prob}(\widehat{N})$ and on the $\pi(N)$-invariant closed subspace $\mathcal{H}_\xi = \overline{\mathrm{Vect}\{\pi(N)\xi\}}$, the strongly continuous unitary subrepresentation $\pi : N \to \mathcal{U}(\mathcal{H}_\xi)$ is unitarily equivalent to the strongly continuous unitary representation $\pi_{\mu_\xi} : N \to \mathcal{U}(\mathrm{L}^2(\widehat{N}, \mu_\xi))$.

Using Riesz's representation theorem, for every $B \in \mathcal{B}(\widehat{N})$, denote by $E_\pi(B) \in \mathbb{B}(\mathcal{H}_\pi)$ the unique bounded operator that satisfies

$$\forall \xi, \eta \in \mathcal{H}_\pi, \quad \langle E_\pi(B)\xi, \eta\rangle = \mu_{\xi,\eta}(B) = \int_{\widehat{N}} \mathbf{1}_B(\chi)\,\mathrm{d}\mu_{\xi,\eta}(\chi). \tag{2.2}$$

By definition, we have $\mu_\xi(B) = \langle E_\pi(B)\xi, \xi\rangle$ for every $B \in \mathcal{B}(\widehat{N})$.

(ii) For every $\xi \in \mathcal{H}_\pi$ with $\|\xi\| = 1$, the bounded operator $E_\pi(B)$ leaves invariant the $\pi(N)$-invariant closed subspace $\mathcal{H}_\xi = \overline{\mathrm{Vect}\{\pi(N)\xi\}}$ and $E_\pi(B)|_{\mathcal{H}_\xi}$ is unitarily equivalent to the bounded operator $\mathrm{L}^2(\widehat{N}, \mu) \to \mathrm{L}^2(\widehat{N}, \mu) : \xi \mapsto \mathbf{1}_B \xi$. This implies that $E_\pi(B)|_{\mathcal{H}_\xi} \in \mathbb{B}(\mathcal{H}_\xi)$ is a selfadjoint projection for every $B \in \mathcal{B}(\widehat{N})$. By Zorn's lemma, there exists a family of unit vectors $(\xi_i)_{i \in I}$ in $\mathcal{H}_\pi$ such that $\mathcal{H}_\pi = \bigoplus_{i \in I} \mathcal{H}_{\xi_i}$. This further implies that $E_\pi(B) \in \mathbb{B}(\mathcal{H}_\pi)$ is a selfadjoint projection for every $B \in \mathcal{B}(\widehat{N})$. For every $\xi \in \mathcal{H}_\pi$ such that $\|\xi\| = 1$, we have $\mu_\xi \in \mathrm{Prob}(\widehat{N})$ and hence

$$\begin{aligned}
\xi \in (\mathcal{H}_\pi)^N \quad &\Leftrightarrow \quad \forall h \in N, \quad \|\pi(h)\xi - \xi\|^2 = 0 \\
&\Leftrightarrow \quad \forall h \in N, \quad \int_{\widehat{N}} |\chi(h) - 1|^2 \, \mathrm{d}\mu_\xi(\chi) = 0 \\
&\Leftrightarrow \quad \mu_\xi = \delta_{1_{\widehat{N}}} \\
&\Leftrightarrow \quad E_\pi(\{1_{\widehat{N}}\})\xi = \xi.
\end{aligned}$$

Uniqueness of the map $E_\pi : \mathcal{B}(\widehat{N}) \to \mathbb{B}(\mathcal{H}_\pi)$ follows from item (i). $\square$

A key step in the proof of Theorem 2.38 is the following intermediate result.

THEOREM 2.40. *Let $\pi : \mathrm{SL}_2(\mathbb{R}) \ltimes \mathbb{R}^2 \to \mathcal{U}(\mathcal{H}_\pi)$ be any strongly continuous unitary representation that contains almost invariant vectors. Then there exists a nonzero $\pi(\mathbb{R}^2)$-invariant vector.*

PROOF. Set $G = \mathrm{SL}_2(\mathbb{R}) \ltimes \mathbb{R}^2$ and $N = \mathbb{R}^2$. For every $g \in \mathrm{SL}_2(\mathbb{R})$ and every $x \in \mathbb{R}^2$, we simply denote by $\mathrm{SL}_2(\mathbb{R}) \times \mathbb{R}^2 \to \mathbb{R}^2 : (g, x) \mapsto g \cdot x$ the action by matrix multiplication. Note that for every $g \in \mathrm{SL}_2(\mathbb{R})$ and every $x \in \mathbb{R}^2$, we have $g \cdot x = gxg^{-1}$ where $G \curvearrowright N$ acts by conjugation. Denote by $(\cdot \mid \cdot)$ the canonical inner product on $\mathbb{R}^2$. We identify the unitary dual of $\mathbb{R}^2$ with $\mathbb{R}^2$ via the following topological isomorphism

$$\mathbb{R}^2 \to \widehat{\mathbb{R}^2} : x \mapsto \widehat{x} = (y \mapsto \exp(\mathrm{i}(x \mid y)))\,.$$

For every $g \in \mathrm{SL}_2(\mathbb{R})$ and every $x \in \mathbb{R}^2$, we then have $\widehat{g \cdot x} = \widehat{g} \cdot \widehat{x}$ where $\widehat{g} = ({}^t g)^{-1}$.

Let $\pi : G \to \mathcal{U}(\mathcal{H}_\pi)$ be any strongly continuous unitary representation such that $1_G \prec \pi$. Then there exists a sequence $(\xi_n)_{n \in \mathbb{N}}$ of $\pi(G)$-almost invariant unit vectors in $\mathcal{H}_\pi$. Applying Theorem 2.39 to $\pi|_{\mathbb{R}^2}$, there exists a sequence $(\mu_{\xi_n})_{n \in \mathbb{N}}$ in $\mathrm{Prob}(\mathbb{R}^2)$ such that

$$\forall n \in \mathbb{N}, \forall x \in \mathbb{R}^2, \quad \langle \pi(x)\xi_n, \xi_n \rangle = \int_{\mathbb{R}^2} \exp(\mathrm{i}(x \mid y)) \, \mathrm{d}\mu_{\xi_n}(y).$$

Then for every $g \in \mathrm{SL}_2(\mathbb{R})$, every $n \in \mathbb{N}$ and every $x \in \mathbb{R}^2$, we have

$$\begin{aligned}
\int_{\mathbb{R}^2} \exp(\mathrm{i}(x \mid y)) \, \mathrm{d}\mu_{\pi(g)\xi_n}(y) &= \langle \pi(x)\pi(g)\xi_n, \pi(g)\xi_n \rangle \\
&= \langle \pi(g^{-1}xg)\xi_n, \xi_n \rangle
\end{aligned}$$

$$
\begin{aligned}
&= \langle \pi(g^{-1} \cdot x)\xi_n, \xi_n \rangle \\
&= \int_{\mathbb{R}^2} \exp(\mathrm{i}(g^{-1} \cdot x \mid y))\, \mathrm{d}\mu_{\xi_n}(y) \\
&= \int_{\mathbb{R}^2} \exp(\mathrm{i}(x \mid \widehat{g} \cdot y))\, \mathrm{d}\mu_{\xi_n}(y) \\
&= \int_{\mathbb{R}^2} \exp(\mathrm{i}(x \mid y))\, \mathrm{d}(\widehat{g}_* \mu_{\xi_n})(y).
\end{aligned}
$$

By uniqueness, it follows that $\mu_{\pi(g)\xi_n} = \widehat{g}_* \mu_{\xi_n}$ for every $g \in G$ and every $n \in \mathbb{N}$. For every $g \in \mathrm{SL}_2(\mathbb{R})$ and every $B \in \mathcal{B}(\mathbb{R}^2)$, we have

$$
\begin{aligned}
|\mu_{\xi_n}({}^t g \cdot B) - \mu_{\xi_n}(B)| &= |(\widehat{g}_* \mu_{\xi_n})(B) - \mu_{\xi_n}(B)| \\
&= |\mu_{\pi(g)\xi_n}(B) - \mu_{\xi_n}(B)| \\
&= |\langle E_\pi(B)\pi(g)\xi_n, \pi(g)\xi_n \rangle - \langle E_\pi(B)\xi_n, \xi_n \rangle| \to 0.
\end{aligned}
$$

Choose a nonprincipal ultrafilter $\mathcal{U} \in \beta(\mathbb{N}) \setminus \mathbb{N}$ and define the map $\mathfrak{m} : \mathcal{B}(\mathbb{R}^2) \to \mathbb{R}_+$ by the formula

$$
\forall B \in \mathcal{B}(\mathbb{R}^2), \quad \mathfrak{m}(B) \doteq \lim_{n \to \mathcal{U}} \mu_{\xi_n}(B).
$$

Then $\mathfrak{m}(\mathbb{R}^2) = 1$, $\mathfrak{m}$ is finitely additive and the above reasoning shows that $\mathfrak{m}(g \cdot B) = \mathfrak{m}(B)$ for every $g \in G$ and every $B \in \mathcal{B}(\mathbb{R}^2)$.

Consider the following Borel partition of $\mathbb{R}^2 \setminus \{(0,0)\}$. Set

$$
\begin{aligned}
V_1 &\doteq \left\{(t_1, t_2) \in \mathbb{R}^2 \mid |t_2| \leq |t_1| \text{ and } t_1 t_2 > 0\right\} \\
V_2 &\doteq \left\{(t_1, t_2) \in \mathbb{R}^2 \mid |t_1| < |t_2| \text{ and } t_1 t_2 \geq 0\right\} \\
V_3 &\doteq \left\{(t_1, t_2) \in \mathbb{R}^2 \mid |t_1| \leq |t_2| \text{ and } t_1 t_2 < 0\right\} \\
V_4 &\doteq \left\{(t_1, t_2) \in \mathbb{R}^2 \mid |t_2| < |t_1| \text{ and } t_1 t_2 \leq 0\right\}.
\end{aligned}
$$

Observe that $\mathbb{R}^2 \setminus \{(0,0)\} = V_1 \sqcup V_2 \sqcup V_3 \sqcup V_4$. Put $g = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Since $g \cdot (V_1 \sqcup V_2) \subset V_1$, we have

$$
\mathfrak{m}(V_1) + \mathfrak{m}(V_2) = \mathfrak{m}(V_1 \sqcup V_2) = \mathfrak{m}(g \cdot (V_1 \sqcup V_2)) \leq \mathfrak{m}(V_1)
$$

and thus $\mathfrak{m}(V_2) = 0$. Similarly, we have $\mathfrak{m}(V_1) = \mathfrak{m}(V_3) = \mathfrak{m}(V_4) = 0$. This implies that $\mathfrak{m} = \delta_{(0,0)}$. This further implies that

$$
\lim_{n \to \mathcal{U}} \langle E_\pi(\{(0,0)\})\xi_n, \xi_n \rangle = \lim_{n \to \mathcal{U}} \mu_n(\{(0,0)\}) = 1.
$$

Therefore $E_\pi(\{(0,0)\}) \neq 0$ and so $\pi|_{\mathbb{R}^2}$ has nonzero invariant vectors. $\square$

We now have all the tools to prove Theorem 2.38.

Proof of Theorem 2.38. Let $d \geq 3$. Regard $\mathrm{SL}_2(\mathbb{R}) \ltimes \mathbb{R}^2 < \mathrm{SL}_d(\mathbb{R})$ as the following subgroup:

$$\mathrm{SL}_2(\mathbb{R}) \ltimes \mathbb{R}^2 \cong \left\{ \begin{pmatrix} A & x & & 0_{3,d-3} \\ 0 & 1 & & \\ & & \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} \\ 0_{d-3,3} & & & \end{pmatrix} \mid A \in \mathrm{SL}_2(\mathbb{R}), x \in \mathbb{R}^2 \right\}.$$

Let $\pi : \mathrm{SL}_d(\mathbb{R}) \to \mathcal{U}(\mathcal{H}_\pi)$ be any strongly continuous unitary representation such that $1_{\mathrm{SL}_d(\mathbb{R})} \prec \pi$. Theorem 2.40 implies that $1_{\mathbb{R}^2} \subset \pi|_{\mathbb{R}^2}$. Since $\mathbb{R}^2$ is not compact, π is not mixing. By Theorem 2.33, we obtain $1_G \subset \pi$. $\square$

Combining Theorems 1.19, 2.38 and Proposition 2.28, we obtain the following corollary.

Corollary 2.41 (Kazhdan [**Ka67**]). *For every $d \geq 3$, $\mathrm{SL}_d(\mathbb{Z})$ has property* (T).

Let us point out that Corollary 2.30 and Theorem 2.38 imply that any lattice $\Gamma < \mathrm{SL}_d(\mathbb{R})$, $d \geq 3$, is weakly uniform. More generally, it is proven in [**Be96**] that any lattice $\Gamma < G$ in a semisimple Lie group G with finite center and no compact factor is weakly uniform.

CHAPTER 3

Stationary measures and Poisson boundaries

In this chapter, we introduce and study the notion of stationary measure. We construct the (G, μ)-Poisson boundary associated with any locally compact group G endowed with an admissible Borel probability measure μ. We then investigate rigidity properties of the (G, μ)-Poisson boundary and its relationship with the notion of amenability.

Introduction

In this chapter, the group G is always assumed to be locally compact second countable. We endow G with its σ-algebra $\mathcal{B}(G)$ of Borel subsets. We fix a left invariant Haar measure m_G on G. Let X be any standard Borel space and denote by $\mathrm{Prob}(X)$ the standard Borel space of all Borel probability measures on X. We say that the action $G \curvearrowright X$ is *Borel* if the action map $\sigma_X : G \times X \to X : (g, x) \mapsto gx$ is Borel. Let $\nu \in \mathrm{Prob}(X)$ and assume that for every $g \in G$, the measures ν and $g_*\nu$ are equivalent on X. In that case, we say that the action $G \curvearrowright (X, \nu)$ is *nonsingular*. Recall that $\mathrm{L}^\infty(X, \nu) = \mathrm{L}^1(X, \nu)^*$ so that $\mathrm{L}^\infty(X, \nu)$ is also endowed with the weak*-topology. By [**Ru91**, Theorem 3.10], we may identify $\mathrm{L}^1(X, \nu)$ with the space of all weak*-continuous linear functionals on $\mathrm{L}^\infty(X, \nu)$. Any nonsingular action $G \curvearrowright (X, \nu)$ gives rise to an action $\alpha : G \curvearrowright \mathrm{L}^\infty(X, \nu)$ defined by the formula

$$\forall g \in G, \forall F \in \mathrm{L}^\infty(X, \nu), \quad \alpha(g)(F) = F \circ g^{-1}.$$

The action map $G \times \mathrm{L}^\infty(X, \nu) \to \mathrm{L}^\infty(X, \nu) : (g, F) \mapsto \alpha(g)(F)$ is separately continuous when $\mathrm{L}^\infty(X, \nu)$ is endowed with the weak*-topology. This follows from the fact that the action $G \curvearrowright \mathrm{L}^1(X, \nu)$ is $\|\cdot\|_1$-continuous. We will then simply say that the action $\alpha : G \curvearrowright \mathrm{L}^\infty(X, \nu)$ is weak*-continuous. We refer the reader to [**Ta03**, Proposition XIII.1.2] for further details. For every Borel probability measure $\eta \in \mathrm{Prob}(X)$ such that $\eta \prec \nu$, me may regard $\eta \in \mathrm{L}^1(X, \nu)$ and we simply denote by $\eta : \mathrm{L}^\infty(X, \nu) \to \mathbb{C} : f \mapsto \int_X f(x)\, \mathrm{d}\eta(x)$ the corresponding weak*-continuous positive unital linear functional. When the context is clear, we will often simply write $\mathrm{L}^\infty(X) = \mathrm{L}^\infty(X, \nu)$.

Recall that we always regard function spaces such as $\mathrm{L}^p(X,\nu)$, for $p \in [1,+\infty]$, over the field $\mathbb{C}$ of complex numbers. The algebra $\mathrm{L}^\infty(X)$ is endowed with the anti-linear involution $*$ defined by $f^*(x) = \overline{f(x)}$ for every $f \in \mathrm{L}^\infty(X)$ and ν-almost every $x \in X$.

Definition 3.1. We say that $\mathcal{A} \subset \mathrm{L}^\infty(X)$ is a *von Neumann subalgebra* if $\mathcal{A}$ is a unital subalgebra of $\mathrm{L}^\infty(X)$ that is stable under the involution $*$ and closed with respect to the weak*-topology.

Remark 3.2. More generally, one can define the notion of von Neumann algebra as follows. Let $\mathcal{H}$ be any complex Hilbert space and consider the unital Banach $*$-algebra $\mathbb{B}(\mathcal{H})$ of all bounded linear operators on $\mathcal{H}$. We say that $M \subset \mathbb{B}(\mathcal{H})$ is a *von Neumann algebra* if M is a unital $*$-subalgebra of $\mathbb{B}(\mathcal{H})$ that is equal to its own bicommutant M'' inside $\mathbb{B}(\mathcal{H})$[1]. By von Neumann's bicommutant theorem, $M \subset \mathbb{B}(\mathcal{H})$ is a von Neumann algebra if and only if M is strongly closed (resp. weakly closed).

For instance, if we view $\mathrm{L}^\infty(X) \subset \mathbb{B}(\mathrm{L}^2(X,\nu))$ as a unital $*$-subalgebra, then one can show that $\mathrm{L}^\infty(X)' = \mathrm{L}^\infty(X)$. In that respect, $\mathrm{L}^\infty(X)$ is an *abelian* von Neumann algebra. Moreover, $\mathcal{A} \subset \mathrm{L}^\infty(X)$ is a von Neumann subalgebra in the sense of Definition 3.1 if and only if $\mathcal{A}$ is equal to its own bicommutant $\mathcal{A}''$ inside $\mathbb{B}(\mathrm{L}^2(X,\nu))$. In these notes, we will only be interested in von Neumann subalgebras $\mathcal{A} \subset \mathrm{L}^\infty(X)$.

When X is a compact metrizable space, we say that the action $G \curvearrowright X$ is *continuous* if the action map $\sigma_X : G \times X \to X : (g,x) \mapsto gx$ is continuous. The next well-known result shows that when dealing with nonsingular actions $G \curvearrowright (X,\nu)$, we may always assume that X is a compact metrizable space and the action $G \curvearrowright X$ is continuous.

Proposition 3.3. *Let $G \curvearrowright (X,\nu)$ be any nonsingular action. Let $\mathcal{A} \subset \mathrm{L}^\infty(X,\nu)$ be any G-invariant von Neumann subalgebra.*

Then there exist a compact metrizable space Z, a continuous action $G \curvearrowright Z$, a measure $\zeta \in \mathrm{Prob}(Z)$ and a G-equivariant measurable factor map $\pi : (X,\nu) \to (Z,\zeta)$ so that the G-equivariant weak*-continuous unital $*$-homomorphism $\pi^* : \mathrm{L}^\infty(Z) \to \mathrm{L}^\infty(X) : F \mapsto F \circ \pi$ satisfies $\nu \circ \pi^* = \zeta$ and

$$\pi^*(\mathrm{L}^\infty(Z)) = \mathcal{A}.$$

For a proof, we refer the reader to [**Ta03**, Proposition XIII.1.2] and [**Zi84**, Proposition B.5, Corollary B.6]. Note that by a G-equivariant measurable factor map $\pi : (X,\nu) \to (Z,\zeta)$, we mean that there exists a ν-conull G-invariant measurable subset $X_0 \subset X$ such that $\pi : X_0 \to Z$ is measurable; for every measurable subset $W \subset Z$, we have $\zeta(W) = \nu(\pi^{-1}(W))$; and for every $g \in G$ and every $x \in X_0$, we have $\pi(gx) = g\pi(x)$.

Applying Proposition 3.3 in the case when $\mathcal{A} = \mathrm{L}^\infty(X)$, we then say that $G \curvearrowright (Z,\zeta)$ is a *compact model* for the nonsingular action $G \curvearrowright (X,\nu)$.

[1]The *commutant* of a subset $S \subset \mathbb{B}(\mathcal{H})$ is defined as $S' = \{T \in \mathbb{B}(\mathcal{H}) \mid \forall S \in S, ST = TS\}$. The *bicommutant* of a subset $S \subset \mathbb{B}(\mathcal{H})$ is defined as $S'' = (S')'$.

In that case, using [Zi84, Corollary B.6], we can further choose $X_0 \subset X$ so that $Z_0 = \pi(X_0) \subset Z$ is a ζ-conull G-invariant measurable subset and $\pi : X_0 \to Z_0$ is bijective and $\pi^{-1} : Z_0 \to X_0$ is also measurable. Thus, up to passing to a compact model, we may assume that $(Z, \zeta) = (X, \nu)$.

Let $G \curvearrowright (X, \nu)$ and $G \curvearrowright (Y, \eta)$ be any nonsingular actions and $\pi : (X, \nu) \to (Y, \eta)$ any G-equivariant measurable factor map. We may regard $\mathrm{L}^\infty(Y) \subset \mathrm{L}^\infty(X)$ as a G-invariant von Neumann subalgebra with $\eta = \nu|_{\mathrm{L}^\infty(X)}$ via the G-equivariant weak*-continuous unital $*$-homomorphism $\pi^* : \mathrm{L}^\infty(Y) \to \mathrm{L}^\infty(X) : F \mapsto F \circ \pi$ such that $\nu \circ \pi^* = \eta$. Recall that there exists a unique conditional expectation $\mathrm{E} : \mathrm{L}^\infty(X) \to \mathrm{L}^\infty(Y)$ such that $\eta \circ \mathrm{E} = \nu$. Note that $\mathrm{E} : \mathrm{L}^\infty(X) \to \mathrm{L}^\infty(Y)$ is *positive* meaning that $\mathrm{E}(f) \geq 0$ for every $f \in \mathrm{L}^\infty(X)$ such that $f \geq 0$. Moreover, $\mathrm{E} : \mathrm{L}^\infty(X) \to \mathrm{L}^\infty(Y)$ is weak*-continuous.

Definition 3.4. We say that the factor map $\pi : (X, \nu) \to (Y, \eta)$ is *relatively measure preserving* if the unique conditional expectation $\mathrm{E} : \mathrm{L}^\infty(X) \to \mathrm{L}^\infty(Y)$ such that $\eta \circ \mathrm{E} = \nu$ is G-equivariant.

1. Stationary measures

Let $\mu \in \mathrm{Prob}(G)$ be any *admissible* Borel probability measure on G, meaning that μ is equivalent to the Haar measure m_G on G. Let $G \curvearrowright X$ be any Borel action on any standard Borel space. Denote by $\sigma_X : G \times X \to X : (g, x) \mapsto gx$ the Borel action map. Let $\nu \in \mathrm{Prob}(X)$ and set $\mu * \nu \doteq \sigma_{X*}(\mu \otimes \nu) \in \mathrm{Prob}(X)$. We say that $\nu \in \mathrm{Prob}(X)$ is *μ-stationary* if $\mu * \nu = \nu$.

Definition 3.5. Keep the same notation as above and assume that $\nu \in \mathrm{Prob}(X)$ is μ-stationary. Then we simply say that (X, ν) is a *(G, μ)-space.*

The first elementary result shows that any stationary measure gives rise to a nonsingular action.

Lemma 3.6. *Let (X, ν) be any (G, μ)-space. Then the action $G \curvearrowright (X, \nu)$ is nonsingular.*

Proof. Let $Y \subset X$ be any measurable subset. Firstly, since the action map $G \times X \to X$ is measurable, the map $G \to \mathbb{C} : h \mapsto \nu(h^{-1}Y)$ is measurable. Secondly, since μ is admissible, we may consider $f \doteq \frac{\mathrm{d}\mu}{\mathrm{d}m_G} \in \mathrm{L}^1(G, m_G)$ with $f \geq 0$ and $\|f\|_1 = 1$. Since the map $G \to \mathrm{L}^1(G, m_G) : h \mapsto \lambda(h)f$ is $\|\cdot\|_1$-continuous and since the measurable map $G \to \mathbb{C} : h \mapsto \nu(h^{-1}Y)$ is bounded, Lebesgue's dominated convergence theorem implies that the map $G \to \mathbb{C} : h \mapsto \nu(h^{-1}Y)$ is continuous because

$$\begin{aligned}\forall h \in G, \quad \nu(h^{-1}Y) &= (\mu * \nu)(h^{-1}Y) \\ &= \int_G \nu(g^{-1}h^{-1}Y) f(g)\, \mathrm{d}m_G(g)\end{aligned}$$

$$= \int_G \nu(g^{-1}Y) f(h^{-1}g)\, \mathrm{d}m_G(g)$$
$$= \int_G \nu(g^{-1}Y)(\lambda(h)f)(g)\, \mathrm{d}m_G(g).$$

Assume that $\nu(Y) = 0$. Then we have

$$0 = \nu(Y) = (\mu * \nu)(Y) = \int_G \nu(g^{-1}Y)\, \mathrm{d}\mu(g).$$

Then for μ-almost every $g \in G$, we have $\nu(g^{-1}Y) = 0$. Since the map $G \to \mathbb{C} : h \mapsto \nu(h^{-1}Y)$ is continuous and since μ is equivalent to the Haar measure m_G on G, it follows that $\nu(g^{-1}Y) = 0$ for every $g \in G$. This shows that the action $G \curvearrowright (X, \nu)$ is nonsingular. $\square$

The second elementary result shows that whenever X is a compact metrizable space and the action $G \curvearrowright X$ is continuous, there always exists at least one μ-stationary measure on X.

Lemma 3.7. *Let X by any compact metrizable G-space. Then there always exists $\nu \in \mathrm{Prob}(X)$ such that $\mu * \nu = \nu$.*

Proof. We define the weak*-continuous affine map $P : \mathrm{Prob}(X) \to \mathrm{Prob}(X)$ on the convex weak*-compact space $\mathrm{Prob}(X)$ by the formula

$$P\nu = \mu * \nu = \int_G g_*\nu\, \mathrm{d}\mu(g).$$

By Markov–Kakutani's fixed point theorem, P has a fixed point $\nu \in \mathrm{Prob}(X)$ which is then a μ-stationary measure. Indeed, let $\eta \in \mathrm{Prob}(X)$ be any measure and define the sequence of measures $(\eta_n)_{n\in\mathbb{N}}$ by the formula

$$\forall n \in \mathbb{N}, \quad \eta_n \doteq \frac{1}{n+1} \sum_{k=0}^{n} P^{\circ n}\eta.$$

Choose a nonprincipal ultrafilter $\mathcal{U} \in \beta(\mathbb{N}) \setminus \mathbb{N}$ and define $\nu = \lim_{n\to\mathcal{U}} \eta_n \in \mathrm{Prob}(X)$ with respect to the weak*-topology. Then we have $P\nu = \nu$ and so $\mu * \nu = \nu$. $\square$

Let (X, ν) be any (G, μ)-space. By Lemma 3.6, the action $G \curvearrowright (X, \nu)$ is nonsingular and so we may consider the weak*-continuous action $\sigma : G \curvearrowright \mathrm{L}^\infty(X)$. We collect functional analytic properties of the (G, μ)-stationary space (X, ν). We refer to [**BBHP20**, Proposition 2.7] for a more general result for arbitrary von Neumann algebras.

Proposition 3.8. *Let (X, ν) be any (G, μ)-space. Denote by $\mathrm{L}^\infty(X)^G \subset \mathrm{L}^\infty(X)$ the von Neumann subalgebra of G-invariant functions and by $\mathrm{E} : \mathrm{L}^\infty(X) \to \mathrm{L}^\infty(X)^G$ the unique conditional expectation such that $\nu \circ \mathrm{E} = \nu$. The following assertions hold:*

(i) *For every $f \in \mathrm{L}^\infty(X)$, $\mathrm{E}(f)$ belongs to the weak*-closure of the convex hull of the set $\{\sigma_g(f) \mid g \in G\}$.*

(ii) *For every μ-stationary Borel probability measure $\eta \in \mathrm{Prob}(X)$ such that $\eta \prec \nu$, we have $\eta \circ \mathrm{E} = \eta$.*

Proof. (i) Recall that $\mathrm{L}^\infty(X) = \mathrm{L}^1(X,\nu)^*$ and define the mapping $T_\mu : \mathrm{L}^\infty(X) \to \mathrm{L}^\infty(X)$ by the formula

$$\forall f \in \mathrm{L}^\infty(X), \forall \psi \in \mathrm{L}^1(X,\nu), \quad \psi(T_\mu(f)) = \int_G \psi(\sigma_g^{-1}(f))\, \mathrm{d}\mu(g).$$

Since $\mu * \nu = \nu$, we have $\nu(T_\mu(f)) = \nu(f)$ for every $f \in \mathrm{L}^\infty(X)$. Observe that since T_μ is positive, this further implies that $T_\mu : \mathrm{L}^\infty(X) \to \mathrm{L}^\infty(X)$ is weak*-continuous. Choose a nonprincipal ultrafilter $\mathcal{U} \in \beta(\mathbb{N}) \setminus \mathbb{N}$. Denote by $\mathrm{E}_\mu : \mathrm{L}^\infty(X) \to \mathrm{L}^\infty(X)$ the mapping defined by the formula

$$\forall f \in \mathrm{L}^\infty(X), \quad \mathrm{E}_\mu(f) \doteq \lim_{n \to \mathcal{U}} \frac{1}{n} \sum_{k=1}^n (T_\mu)^{\circ k}(f)$$

where the above limit is taken with respect to the weak*-topology. For every $f \in \mathrm{L}^\infty(X)$, we have $\nu(\mathrm{E}_\mu(f)) = \nu(f)$ and $\mathrm{E}_\mu(f) \in \mathrm{L}^\infty(X)^G$. Indeed, set $a = \mathrm{E}_\mu(f) \in \mathrm{L}^\infty(X)$. Then we have $T_\mu(a) = a$ and so

$$\begin{aligned}\int_G \|a - \sigma_g^{-1}(a)\|_\nu^2 \,\mathrm{d}\mu(g) &= \nu(a^*a) - 2\Re(\nu(aT_\mu(a))) + \nu(T_\mu(a^*a)) \\ &= \nu(a^*a) - 2\Re(\nu(a^*a)) + \nu(a^*a) \\ &= 0.\end{aligned}$$

This implies that $a = \sigma_g^{-1}(a)$ for μ-almost every $g \in G$. Since μ is equivalent to the Haar measure on G and since the map $G \to \mathrm{L}^\infty(X) : g \mapsto \sigma_g^{-1}(a)$ is weak*-continuous, it follows that $\sigma_g(a) = a$ for every $g \in G$. Thus, $\mathrm{E}_\mu : \mathrm{L}^\infty(X) \to \mathrm{L}^\infty(X)^G$ is the unique conditional expectation such that $\nu \circ \mathrm{E}_\mu = \nu$.

Denote by $\mathcal{C} \subset \mathrm{L}^\infty(X)$ the weak*-closure of the convex hull of the set $\{\sigma_g(f) \mid g \in G\}$. By construction and using Hahn–Banach theorem, for every $f \in \mathrm{L}^\infty(X)$, we have $T_\mu(f) \in \mathcal{C}$. Indeed, otherwise using [**Ru91**, Theorem 3.4(b)], there would exist $\psi \in \mathrm{L}^1(X,\nu)$ and $\alpha \in \mathbb{R}$ such that

$$\forall g \in G, \quad \Re\left(T_\mu(\psi(f))\right) < \alpha \leq \Re\left(\psi(\sigma_{g^{-1}}(f))\right).$$

This would imply that

$$\Re\left(T_\mu(\psi(f))\right) = \int_G \Re\left(\psi(\sigma_{g^{-1}}(f))\right) \mathrm{d}\mu(g) \geq \alpha > \Re\left(T_\mu(\psi(f))\right),$$

a contradiction. Then for every $f \in \mathrm{L}^\infty(X)$, we have $T_\mu(f) \in \mathcal{C}$ and hence

$$\mathrm{E}_\mu(f) = \text{w-}^* \lim_{n \to \mathcal{U}} \frac{1}{n} \sum_{k=1}^n (T_\mu)^{\circ k}(f) \in \mathcal{C}.$$

(ii) Let $\eta \in \mathrm{Prob}(X)$ be any μ-stationary Borel probability measure such that $\eta \prec \nu$ and regard $\eta \in \mathrm{L}^1(X,\nu)$. Since η is μ-stationary, we have $\eta \circ T_\mu = \eta$. This further implies that $\eta \circ \mathrm{E}_\mu = \eta$. $\square$

The third elementary result deals with the equivalence between extremality of the stationary measure and ergodicity of the associated nonsingular action. In that respect, we say that a compact metrizable (G,μ)-space (X,ν) is *extremal* if $\nu \in \mathrm{Prob}(X)$ is an extremal point in the convex weak*-compact subset $\mathrm{Prob}_\mu(X)$ of all μ-stationary Borel probability measures in $\mathrm{Prob}(X)$. We say that a nonsingular action $G \curvearrowright (X,\nu)$ is *ergodic* if every G-invariant measurable subset $Y \subset X$ is null or conull. Observe that the nonsingular action $G \curvearrowright (X,\nu)$ is ergodic if and only if $\mathrm{L}^\infty(X)^G = \mathbb{C}\mathbf{1}_X$ (see the proof of Proposition 2.7).

LEMMA 3.9. *Let (X,ν) be any compact metrizable (G,μ)-space. The following assertions are equivalent:*

(i) *The (G,μ)-space (X,ν) is extremal.*
(ii) *The nonsingular action $G \curvearrowright (X,\nu)$ is ergodic.*

PROOF. (i) $\Rightarrow$ (ii) By contraposition, assume that the nonsingular action $G \curvearrowright (X,\nu)$ is not ergodic. Choose a G-invariant measurable subset $Y \subset X$ such that $0 < \nu(Y) < 1$. Define $\nu_1 \in \mathrm{Prob}_\mu(X)$ by $\nu_1 = \frac{1}{\nu(Y)}\nu|_Y$ and $\nu_2 \in \mathrm{Prob}_\mu(X)$ by $\nu_2 = \frac{1}{\nu(Y^c)}\nu|_{Y^c}$. Then $\nu = \alpha\nu_1 + (1-\alpha)\nu_2$ with $\alpha = \nu(Y) > 0$ and $\nu \neq \nu_1, \nu_2$. Therefore, the (G,μ)-space (X,ν) is not extremal.

(ii) $\Rightarrow$ (i) Since the nonsingular action $G \curvearrowright (X,\nu)$ is ergodic, we have $\mathrm{L}^\infty(X)^G = \mathbb{C}\mathbf{1}_X$. Proposition 3.8 implies that $\mathrm{E}_\mu(f) = \nu(f)\mathbf{1}_X$ for every $f \in \mathrm{L}^\infty(X)$. Assume that $\nu = \alpha\nu_1 + (1-\alpha)\nu_2$ with $\alpha > 0$ and $\nu_1, \nu_2 \in \mathrm{Prob}_\mu(X)$. Since $\nu_1 \leq \frac{1}{\alpha}\nu$, we have $\nu_1 \prec \nu$. Proposition 3.8(iii) implies that $\nu_1(f) = \nu_1(\mathrm{E}_\mu(f)) = \nu_1(\nu(f)\mathbf{1}_X) = \nu(f)$ for every $f \in \mathrm{L}^\infty(X)$ and so $\nu_1 = \nu$. Likewise, we have $\nu_2 = \nu$. This shows that the (G,μ)-space (X,ν) is extremal. ☐

The fourth elementary result deals with stationary measures supported on countable sets. Whenever X is a compact metrizable space G-space and $\nu \in \mathrm{Prob}(X)$, we denote by $\mathrm{supp}(\nu) \subset X$ the *topological support* of the measure ν. By definition, $\mathrm{supp}(\nu)$ is the intersection of all closed subsets $Y \subset X$ for which $\nu(Y) = 1$. Then $\mathrm{supp}(\nu) \subset X$ is closed and $\nu(\mathrm{supp}(\nu)) = 1$.

LEMMA 3.10. *Let (X,ν) be any extremal compact metrizable (G,μ)-space. Assume that ν has an atom. Then ν is G-invariant and $\mathrm{supp}(\nu) \subset X$ is a finite set.*

PROOF. Choose $x \in \mathrm{supp}(\nu)$ an atom of maximum mass. Since

$$\nu(\{x\}) = \int_G g_*\nu(\{x\})\,\mathrm{d}\mu(g) = \int_G \nu(\{g^{-1}x\})\,\mathrm{d}\mu(g),$$

it follows that $\nu(\{g^{-1}x\}) = \nu(\{x\})$ for μ-almost every $g \in G$. Since $\mu \in \mathrm{Prob}(G)$ is admissible, the map $G \to \mathbb{C} : g \mapsto \nu(\{g^{-1}x\})$ is continuous (see the proof of Lemma 3.6) and hence we have $\nu(\{g^{-1}x\}) = \nu(\{x\})$ for every $g \in G$. Therefore, Gx is finite and $\frac{1}{\nu(Gx)}\nu|_{Gx}$ is a G-invariant finitely supported probability measure. Since ν is assumed to be extremal among μ-stationary measures, it follows that $\nu = \frac{1}{\nu(Gx)}\nu|_{Gx}$ is G-invariant and supported on Gx. $\square$

Denote by $\mathrm{B}(G)$ the unital $*$-algebra of all bounded Borel functions on G. Define the *Markov operator* $P_\mu : \mathrm{B}(G) \to \mathrm{B}(G)$ by

$$\forall g \in G, \quad P_\mu(F)(g) \doteq \int_G F(gh)\,\mathrm{d}\mu(h).$$

Observe that P_μ is a unital positive linear contraction. Following [**Fu62a**], a function $F \in \mathrm{B}(G)$ is said to be (right) *μ-harmonic* if $P_\mu(F) = F$. We denote by $\mathrm{Har}(G, \mu) = \ker(P_\mu - \mathrm{id})$ the space of all bounded (right) μ-harmonic functions. The next result shows that all bounded μ-harmonic functions are continuous.

LEMMA 3.11. *We have* $\mathrm{Har}(G, \mu) \subset \mathrm{C}_b(G)$.

PROOF. Since μ is admissible, we may consider $f \doteq \frac{\mathrm{d}\mu}{\mathrm{d}m_G} \in \mathrm{L}^1(G, m_G)$ with $f \geq 0$ and $\|f\|_1 = 1$. Recall that the map $G \to \mathrm{L}^1(G, m_G) : h \mapsto \lambda(h)f$ is $\|\cdot\|_1$-continuous. For every $F \in \mathrm{Har}(G, \mu)$, we have

$$\begin{aligned}
F(g) &= \int_G F(gh)\,\mathrm{d}\mu(h) \\
&= \int_G F(gh)f(h)\,\mathrm{d}m_G(h) \\
&= \int_G F(h)f(g^{-1}h)\,\mathrm{d}m_G(h) \\
&= \int_G F(h)(\lambda(g)f)(h)\,\mathrm{d}m_G(h).
\end{aligned}$$

Since F is uniformly bounded, Lebesgue's dominated convergence theorem implies that F is continuous. Thus, $\mathrm{Har}(G, \mu) \subset \mathrm{C}_b(G)$. $\square$

Let (X, ν) be any (G, μ)-space. Denote by $\mathrm{B}(X)$ the unital $*$-algebra of all bounded Borel functions on X. Define the *Poisson transform* $\Phi_\mu : \mathrm{B}(X) \to \mathrm{Har}(G, \mu)$ by the formula

$$\forall g \in G, \quad \Phi_\mu(f)(g) \doteq \int_X f(gx)\,\mathrm{d}\nu(x).$$

The function $F = \Phi_\mu(f)$ is indeed μ-harmonic, since by Fubini's theorem, we have

$$\begin{aligned}
\int_G F(gh)\,\mathrm{d}\mu(h) &= \int_G \left(\int_X f(ghx)\,\mathrm{d}\nu(x)\right)\mathrm{d}\mu(h) \\
&= \int_{G\times X} f(g\,\sigma_X(h,x))\,\mathrm{d}(\mu\otimes\nu)(h,x) \\
&= \int_X f(gy)\,\mathrm{d}(\mu * \nu)(y)
\end{aligned}$$

$$= \int_X f(gy)\, \mathrm{d}\nu(y) = F(g).$$

Observe that Φ_μ is a G-equivariant unital positive linear contraction.

2. The limit probability measures

The main result of this section provides the existence of limit probability measures associated with any stationary measure.

Let $\mu \in \operatorname{Prob}(G)$ be any admissible Borel probability measure. Set $(\Omega, \mathcal{F}, \mathbf{P}) = (G^{\mathbb{N}}, \mathcal{B}(G)^{\otimes \mathbb{N}}, \mu^{\otimes \mathbb{N}})$. Define the *forward shift* $S : \Omega \to \Omega$ by the formula

$$\forall (g_n)_{n \in \mathbb{N}} \in \Omega, \quad S((g_n)_{n \in \mathbb{N}}) \doteq (g_{n+1})_{n \in \mathbb{N}}.$$

Observe that $S_*\mathbf{P} = \mathbf{P}$ and moreover S is $\mathbf{P}$-ergodic. We simply write $\omega = (g_n)_{n \in \mathbb{N}} \in \Omega$.

THEOREM 3.12 (Furstenberg [Fu62b]). *Let (X, ν) be a compact metrizable (G, μ)-space. Then there exists a measurable map $\Omega \to \operatorname{Prob}(X) : \omega \mapsto \nu_\omega$ that satisfies the following properties:*

(i) *For $\mathbf{P}$-almost every $\omega = (g_n)_{n \in \mathbb{N}} \in \Omega$, the sequence $(g_{0*} \cdots g_{n*}\nu)_{n \in \mathbb{N}}$ converges to $\nu_\omega \in \operatorname{Prob}(X)$ with respect to the weak*-topology.*

(ii) *For $\mathbf{P}$-almost every $\omega = (g_n)_{n \in \mathbb{N}} \in \Omega$ and for μ-almost every $g \in G$, the sequence $(g_{0*} \cdots g_{n*} g_* \nu)_{n \in \mathbb{N}}$ still converges to $\nu_\omega \in \operatorname{Prob}(X)$ with respect to the weak*-topology.*

(iii) *For $\mathbf{P}$-almost every $\omega = (g_n)_{n \in \mathbb{N}} \in \Omega$, we have $\nu_\omega = g_{0*}\nu_{S(\omega)}$ and*

$$\nu = \int_\Omega \nu_\omega \, \mathrm{d}\mathbf{P}(\omega).$$

PROOF. (i) For every $f \in \mathrm{C}(X)$, we have

$$(g_{0*} \cdots g_{n*}\nu)(f) = \int_X f(g_0 \cdots g_n x)\, \mathrm{d}\nu(x) = \Phi_\mu(f)(g_0 \cdots g_n).$$

Define the uniformly bounded sequence $F_n \in \mathrm{L}^\infty(\Omega, \mathbf{P})$ by the formula

$$\forall \omega = (g_n)_{n \in \mathbb{N}} \in \Omega, \quad F_n(\omega) \doteq \Phi_\mu(f)(g_0 \cdots g_n).$$

Define the increasing sequence of σ-subalgebras $\mathcal{F}_n \subset \mathcal{F}$ by the formula $\mathcal{F}_n = \sigma(X_0, \dots, X_n)$ where $X_n : \Omega \to G : \omega \mapsto g_n$ is the projection onto the n-th coordinate for all $n \in \mathbb{N}$. Observe that for every $n \in \mathbb{N}$, $F_n \in \mathrm{L}^\infty(\Omega, \mathcal{F}_n, \mathbf{P})$ with $\|F_n\|_\infty \leq \|f\|_\infty$ and $\bigvee_{n \in \mathbb{N}} \mathcal{F}_n = \mathcal{F}$. A simple calculation using μ-harmonicity shows that

$$\begin{aligned}
\forall \omega = (g_n)_{n \in \mathbb{N}} \in \Omega, \quad \mathbf{E}\left[F_{n+1} \mid \mathcal{F}_n\right](\omega) &= \int_G \Phi_\mu(f)(g_0 \cdots g_n g'_{n+1})\, \mathrm{d}\mu(g'_{n+1}) \\
&= \Phi_\mu(f)(g_0 \cdots g_n) \\
&= F_n(\omega).
\end{aligned}$$

It follows that $(F_n)_{n \in \mathbb{N}}$ is a uniformly bounded martingale, hence it converges $\mathbf{P}$-almost everywhere. Set $F(\omega) = \lim_n F_n(\omega)$ for $\mathbf{P}$-almost every $\omega \in \Omega$.

Since X is a compact metrizable space, $\mathrm{C}(X)$ is separable with respect to the uniform norm. Choose a uniformly dense countable subset $\mathcal{A} \subset \mathrm{C}(X)$. Let $\Omega_0 \subset \Omega$ be a Borel subset such that $\mathbf{P}(\Omega_0) = 1$ and such that for every $\omega = (g_n)_{n\in\mathbb{N}} \in \Omega_0$, we have that $(g_{0*} \cdots g_{n*}\nu)(f)$ is convergent for all $f \in \mathcal{A}$. For every $\omega = (g_n)_{n\in\mathbb{N}} \in \Omega_0$, define the bounded mapping

$$\mathcal{A} \to \mathbb{C} : f \mapsto \lim_n (g_{0*} \cdots g_{n*}\nu)(f).$$

This mapping extends uniquely to a positive norm one bounded linear functional

$$\mathrm{C}(X) \to \mathbb{C} : f \mapsto \lim_n (g_{0*} \cdots g_{n*}\nu)(f).$$

Hence, by Riesz representation theorem, for every $\omega = (g_n)_{n\in\mathbb{N}} \in \Omega_0$, there is a unique Borel probability measure $\nu_\omega \in \mathrm{Prob}(X)$ such that $g_{0*} \cdots g_{n*}\nu \to \nu_\omega$ with respect to the weak*-topology. We can then define a measurable map $\Omega \to \mathrm{Prob}(X) : \omega \mapsto \nu_\omega$ such that for $\mathbf{P}$-almost every $\omega \in \Omega$, we have $g_{0*} \cdots g_{n*}\nu \to \nu_\omega$ with respect to the weak*-topology.

(ii) Let $f \in \mathrm{C}(X)$. For every $g \in G$, define $F_n^g \in \mathrm{L}^\infty(\Omega, \mathbf{P})$ by $F_n^g(\omega) = \Phi_\mu(f)(g_0 \cdots g_n g)$. For every $n \in \mathbb{N}$, let us define and compute

$$\begin{aligned} I_n &\doteq \int_\Omega \int_G |F_n(\omega) - F_n^g(\omega)|^2 \,\mathrm{d}\mu(g)\,\mathrm{d}\mathbf{P}(\omega) \\ &= \int_G \int_G |\Phi_\mu(f)(h) - \Phi_\mu(f)(hg)|^2 \,\mathrm{d}\mu(g)\,\mathrm{d}\mu^{*(n+1)}(h) \\ &= \int_\Omega \int_G |F_n(\omega) - F_{n+1}(\omega)|^2 \,\mathrm{d}\mathbf{P}(\omega) \\ &= \|F_n - F_{n+1}\|^2_{\mathrm{L}^2(\Omega,\mathbf{P})}. \end{aligned}$$

Since $(F_n)_{n\in\mathbb{N}}$ is a martingale, we have

$$I_n = \|F_{n+1}\|^2_{\mathrm{L}^2(\Omega,\mathbf{P})} - \|F_n\|^2_{\mathrm{L}^2(\Omega,\mathbf{P})}.$$

This implies that $\sum_{n\in\mathbb{N}} I_n \leq \|f\|_\infty^2$ and hence

$$\left((\omega, g) \mapsto \sum_{n\in\mathbb{N}} |F_n(\omega) - F_n^g(\omega)|^2\right) \in \mathrm{L}^1(\Omega \times G, \mathbf{P} \otimes \mu).$$

In particular, for $\mathbf{P}$-almost every $\omega \in \Omega$ and μ-almost every $g \in G$, we have $\lim_n (F_n(\omega) - F_n^g(\omega)) = 0$. This shows that for $\mathbf{P}$-almost every $\omega \in \Omega$ and μ-almost every $g \in G$, we have

$$\lim_n (g_{0*} \cdots g_{n*}\nu)(f) = \nu_\omega(f) = \lim_n (g_{0*} \cdots g_{n*} g_* \nu)(f).$$

This implies that for $\mathbf{P}$-almost every $\omega \in \Omega$ and μ-almost every $g \in G$, we have

$$\lim_n g_{0*} \cdots g_{n*} g_* \nu = \nu_\omega.$$

(iii) For $\mathbf{P}$-almost every $\omega = (g_n)_{n\in\mathbb{N}} \in \Omega$, we have

$$g_{0*}\nu_{S(\omega)} = \lim_n g_{0*} g_{1*} \cdots g_{n*}\nu = \nu_\omega$$

with respect to the weak*-topology. Moreover, for every $f \in \mathrm{C}(X)$, using again μ-harmonicity and Lebesgue's dominated convergence theorem, we have

$$\begin{aligned}\int_\Omega \nu_\omega(f)\,\mathrm{d}\mathbf{P}(\omega) &= \int_\Omega \lim_n (g_{0*}\cdots g_{n*}\nu)(f)\,\mathrm{d}\mathbf{P}(\omega)\\ &= \int_\Omega \lim_n \Phi_\mu(f)(g_0\cdots g_n)\,\mathrm{d}\mathbf{P}(\omega)\\ &= \lim_n \int_\Omega \Phi_\mu(f)(g_0\cdots g_n)\,\mathrm{d}\mathbf{P}(\omega)\\ &= \lim_n \Phi_\mu(f)(e)\\ &= \int_X f(x)\,\mathrm{d}\nu(x).\end{aligned}$$

This implies that $\nu = \int_\Omega \nu_\omega\,\mathrm{d}\mathbf{P}(\omega)$. □

Remark 3.13. Let (X, ν) be any compact metrizable (G, μ)-space. We point out that the integral formula in Theorem 3.12(iii) can be upgraded to hold for all bounded Borel functions on X. More precisely, for every $f \in \mathrm{B}(X)$, the map $\Omega \to \mathbb{C} : \omega \mapsto \nu_\omega(f)$ is measurable and we have

$$\nu(f) = \int_\Omega \nu_\omega(f)\,\mathrm{d}\mathbf{P}(\omega).$$

We refer to [**NZ00**, Lemma 2.2] for a proof of this fact.

Any G-invariant measure is necessarily μ-stationary. The converse holds when the group G is abelian.

Theorem 3.14 (Choquet–Deny [**CD60**]). *Let G be any abelian locally compact secound countable group and (X, ν) any compact metrizable (G, μ)-space. Then ν is G-invariant.*

Proof. Let S_∞ be the countable discrete group of finitely supported permutations of $\mathbb{N}$. Define the Borel pmp action $S_\infty \curvearrowright (\Omega, \mathbf{P})$ by

$$\sigma \cdot ((g_n)_{n\in\mathbb{N}}) = (g_{\sigma^{-1}(n)})_{n\in\mathbb{N}}.$$

By the Hewitt–Savage zero-one law (see [**HS53**]), the action $S_\infty \curvearrowright (\Omega, \mathbf{P})$ is ergodic. Since G is abelian, Theorem 3.12(i) implies that $\nu_\omega = \nu_{\sigma(\omega)}$ for every $\sigma \in S_\infty$ and $\mathbf{P}$-almost every $\omega \in \Omega$. By ergodicity and since X is a compact metrizable space, the measurable function $\Omega \to \mathrm{Prob}(X) : \omega \mapsto \nu_\omega$ is $\mathbf{P}$-almost everywhere constant and hence equal to ν by Theorem 3.12(iii). Since G is abelian, Theorem 3.12(ii) implies that $g_*\nu = \nu$ for μ-almost every $g \in G$. Since $G \curvearrowright X$ is continuous, the action $G \curvearrowright \mathrm{Prob}(X)$ is weak*-continuous. Since μ is equivalent to the Haar measure, we conclude that ν is G-invariant. □

3. The Poisson boundary

In this section, we construct the Poisson boundary associated with an admissible measure $\mu \in \mathrm{Prob}(G)$. As we will see, the Poisson boundary is the (essentially) unique (G, μ)-space (B, ν_B) for which the Poisson transform $\Phi_\mu : \mathrm{L}^\infty(B, \nu_B) \to \mathrm{Har}(G, \mu)$ is surjective and isometric. We follow the exposition given in [**BS04**, §2].

As in the previous section, set $(\Omega, \mathcal{F}, \mathbf{P}) = (G^{\mathbb{N}}, \mathcal{B}(G)^{\otimes \mathbb{N}}, \mu^{\otimes \mathbb{N}})$. Define the Borel action $G \curvearrowright \Omega$ by the formula

$$\forall g \in G, \forall \omega = (g_n)_{n \in \mathbb{N}} \in \Omega, \quad g \cdot (g_0, g_1, \dots) = (g g_0, g_1, \dots).$$

Observe that the action $G \curvearrowright \Omega$ is moreover nonsingular. Indeed, for every $g \in G$, we have $g_* \mathbf{P} = g_* \mu \otimes \mathbf{P}^*$ and $\mathbf{P} = \mu \otimes \mathbf{P}^*$ with $\mathbf{P}^* = \prod_{n \geq 1}^{\otimes} \mu$. Since μ is equivalent to the Haar measure m_G on G, we have $g_* \mu \sim \mu$ and so $g_* \mathbf{P} \sim \mathbf{P}$. This gives rise to a weak*-continuous action $\alpha : G \curvearrowright \mathrm{L}^\infty(\Omega, \mathbf{P})$. Likewise, define the nonsingular transformation $T : (\Omega, \mathbf{P}) \to (\Omega, \mathbf{P})$ by $T(g_0, g_1, \dots) = (g_0 g_1, g_2, \dots)$. Indeed, we have $T_* \mathbf{P} = \mu^{*2} \otimes \prod_{n \geq 2}^{\otimes} \mu$ and $\mu^{*2} \sim \mu$ so that $T_* \mathbf{P} \sim \mathbf{P}$. Moreover, we have $T \circ g = g \circ T$ for all $g \in G$. Set

$$\mathrm{L}^\infty(\Omega, \mathbf{P})^T = \{F \in \mathrm{L}^\infty(\Omega, \mathbf{P}) \mid F \circ T = F\}.$$

Since the nonsingular transformation T and the nonsingular action of $G \curvearrowright (\Omega, \mathbf{P})$ commute, $\mathrm{L}^\infty(\Omega, \mathbf{P})^T \subset \mathrm{L}^\infty(\Omega, \mathbf{P})$ is a G-invariant von Neumann subalgebra. Then Proposition 3.3 implies that there exist a standard probability space (B, ν_B), a nonsingular action $G \curvearrowright (B, \nu_B)$ and a G-equivariant measurable factor map $\pi : (\Omega, \mathbf{P}) \to (B, \nu_B)$ so that the mapping $\pi^* : \mathrm{L}^\infty(B) \to \mathrm{L}^\infty(\Omega)^T : f \mapsto f \circ \pi$ is a G-equivariant weak*-continuous unital $*$-isomorphism such that $\mathbf{P} \circ \pi^* = \nu_B$.

Alternatively, we can define the space (B, ν_B) as follows. For every $n \in \mathbb{N}$, define the random variable $W_n : \Omega \to G : \omega \mapsto g_0 \cdots g_n$. Then define the *tail* σ-algebra

$$\mathcal{T} \doteq \bigcap_{n \in \mathbb{N}} \sigma(W_n, W_{n+1}, \dots) \subset \mathcal{F}.$$

Then we have $\mathrm{L}^\infty(B, \nu_B) = \mathrm{L}^\infty(\Omega, \mathcal{T}, \mathbf{P}) \subset \mathrm{L}^\infty(\Omega, \mathcal{F}, \mathbf{P})$.

Claim 3.15. The measure $\nu_B \in \mathrm{Prob}(B)$ is μ-stationary.

Indeed, denote by $\sigma_\Omega : G \times \Omega \to \Omega$ and $\sigma_B : G \times B \to B$ the Borel maps given by the nonsingular actions $G \curvearrowright \Omega$ and $G \curvearrowright B$. By definition of the G-equivariant factor map π, we have $\pi \circ T = \pi$ and $\pi \circ \sigma_\Omega = \sigma_B \circ (\mathrm{id}_G \times \pi)$. Moreover, we have $T_* \mathbf{P} = \sigma_{\Omega *}(\mu \otimes \mathbf{P})$. Therefore, we obtain

$$\begin{aligned} \nu_B &= \pi_* \mathbf{P} = (\pi \circ T)_* \mathbf{P} = \pi_*(T_* \mathbf{P}) \\ &= \pi_*(\sigma_{\Omega *}(\mu \otimes \mathbf{P})) = (\pi \circ \sigma_\Omega)_*(\mu \otimes \mathbf{P}) = \sigma_{B *}(\mu \otimes \nu_B) \\ &= \mu * \nu_B. \end{aligned}$$

From now on, we use the identification $\mathrm{L}^\infty(\Omega, \mathbf{P})^T = \mathrm{L}^\infty(B, \nu_B)$ with $\nu_B = \mathbf{P}|_{\mathrm{L}^\infty(\Omega)^T}$. We will simply write $\overline{\omega} \in B$ for the image of $\omega \in \Omega$ in B. Claim 3.15 shows that (B, ν_B) is a (G, μ)-space. We will prove that for $\mathbf{P}$-almost every $\omega \in \Omega$, the sequence $(W_n(\omega))_{n\in\mathbb{N}} = (g_0 \cdots g_n)_{n\in\mathbb{N}}$ converges "in a certain sense" towards the point $\overline{\omega} \in B$ (see (3.1)).

THEOREM 3.16 (Furstenberg [**Fu62b**]). *The Poisson transform*

$$\Phi_\mu : \mathrm{L}^\infty(B, \nu_B) \to \mathrm{Har}(G, \mu) : f \mapsto \left(g \mapsto \int_B f(gb) \, \mathrm{d}\nu_B(b)\right)$$

is a G-equivariant unital positive surjective linear isometry.

PROOF. We know that $\Phi_\mu : \mathrm{L}^\infty(B, \nu_B) \to \mathrm{Har}(G, \mu)$ is a G-equivariant unital positive linear contraction. It remains to construct the inverse map of Φ_μ. Let $F \in \mathrm{Har}(G, \mu)$ and define the sequence $(\widehat{F}_n)_{n\in\mathbb{N}}$ in $\mathrm{L}^\infty(\Omega, \mathcal{F}, \mathbf{P})$ by the formula

$$\forall \omega = (g_n)_{n\in\mathbb{N}} \in \Omega, \quad \widehat{F}_n(\omega) = F(g_0 \cdots g_n).$$

Define the increasing sequence of σ-subalgebras $\mathcal{F}_n \subset \mathcal{F}$ by the formula $\mathcal{F}_n = \sigma(X_0, \dots, X_n)$ where $X_n : \Omega \to G : \omega \mapsto g_n$ is the projection onto the n-th coordinate for all $n \in \mathbb{N}$. Observe that for every $n \in \mathbb{N}$, $\widehat{F}_n \in \mathrm{L}^\infty(\Omega, \mathcal{F}_n, \mathbf{P})$ with $\|\widehat{F}_n\|_\infty \leq \|F\|_\infty$ and $\bigvee_{n\in\mathbb{N}} \mathcal{F}_n = \mathcal{F}$. A simple calculation using μ-harmonicity shows that

$$\begin{aligned}
\forall \omega = (g_n)_{n\in\mathbb{N}}, \quad \mathbf{E}\left[\widehat{F}_{n+1} \mid \mathcal{F}_n\right](\omega) &= \int_G F(g_0 g_1 \cdots g_n g'_{n+1}) \, \mathrm{d}\mu(g'_{n+1}) \\
&= F(g_0 g_1 \cdots g_n) \\
&= \widehat{F}_n(\omega).
\end{aligned}$$

Thus, $(\widehat{F}_n)_{n\in\mathbb{N}}$ is a uniformly bounded martingale, hence it converges $\mathbf{P}$-almost everywhere. Set $\widehat{F}(\omega) \doteq \lim_n \widehat{F}_n(\omega)$ for $\mathbf{P}$-almost every $\omega \in \Omega$. It follows that $\widehat{F} \in \mathrm{L}^\infty(\Omega, \mathbf{P})$ with $\|\widehat{F}\|_\infty \leq \|F\|_\infty$. Moreover, we have

$$(\widehat{F}_n \circ T)(g_0, g_1, \dots) = \widehat{F}_n(g_0 g_1, \dots) = F(g_0 g_1 \cdots g_n g_{n+1}) = \widehat{F}_{n+1}(\omega).$$

Therefore $\widehat{F} \circ T = \widehat{F}$ and so $\widehat{F} \in \mathrm{L}^\infty(\Omega, \mathbf{P})^T = \mathrm{L}^\infty(B, \nu_B)$.

The map $\Psi_\mu : \mathrm{Har}(G, \mu) \to \mathrm{L}^\infty(B, \nu_B) : F \mapsto \widehat{F}$ is a G-equivariant unital positive linear contraction. It remains to prove that Ψ_μ is indeed an inverse for Φ_μ. If $F \in \mathrm{Har}(G, \mu)$, using Lebesgue's dominated convergence theorem and regarding $\widehat{F} \in \mathrm{L}^\infty(B, \nu_B) = \mathrm{L}^\infty(\Omega, \mathbf{P})^T$, for every $g \in G$, we obtain

$$\begin{aligned}
\Phi_\mu(\widehat{F})(g) &= \int_B \widehat{F}(g\overline{\omega}) \, \mathrm{d}\nu_B(\overline{\omega}) \\
&= \int_\Omega \widehat{F}(g \cdot \omega) \, \mathrm{d}\mathbf{P}(\omega) \\
&= \lim_n \int_\Omega \widehat{F}_n(g \cdot \omega) \, \mathrm{d}\mathbf{P}(\omega)
\end{aligned}$$

$$
\begin{aligned}
&= \lim_n \int_\Omega F(gg_0 \cdots g_n)\,\mathrm{d}\mathbf{P}(\omega) \\
&= \lim_n F(g) = F(g).
\end{aligned}
$$

Conversely, let $f \in \mathrm{L}^\infty(B, \nu_B) = \mathrm{L}^\infty(\Omega, \mathbf{P})^T$. Then for every $n \in \mathbb{N}$ and for $\mathbf{P}$-almost every $\omega \in \Omega$, we have

$$
\begin{aligned}
\mathbf{E}\left[\widehat{\Phi_\mu(f)} \mid \mathcal{F}_n\right](\omega) &= \widehat{\Phi_\mu(f)}_n(\omega) \\
&= \Phi_\mu(f)(g_0 \cdots g_n) \\
&= \int_B f(g_0 \cdots g_n \overline{\omega}')\,\mathrm{d}\nu_B(\overline{\omega}') \\
&= \int_\Omega f(g_0 \cdots g_n \cdot \omega')\,\mathrm{d}\mathbf{P}(\omega') \\
&= \int_\Omega f \circ T^{n+1}(g_0, \dots, g_n, \omega')\,\mathrm{d}\mathbf{P}(\omega') \\
&= \int_\Omega f(g_0, \dots, g_n, \omega')\,\mathrm{d}\mathbf{P}(\omega') \\
&= \mathbf{E}\left[f \mid \mathcal{F}_n\right](\omega).
\end{aligned}
$$

It follows that $\widehat{\Phi_\mu(f)} = f$. Therefore, the Poisson transform

$$\Phi_\mu : \mathrm{L}^\infty(B, \nu_B) \to \mathrm{Har}(G, \mu)$$

and the mapping

$$\Psi_\mu : \mathrm{Har}(G, \mu) \to \mathrm{L}^\infty(B, \nu_B) : F \mapsto \widehat{F}$$

are inverse of one another. Moreover, for every $f \in \mathrm{Har}(G, \mu)$, we have $\|\Phi_\mu(f)\|_\infty = \|f\|_\infty$. □

DEFINITION 3.17. The (G, μ)-space (B, ν_B) is called the (G, μ)-*Poisson boundary.*

Even though we will not use it, we state a fundamental result due to Furstenberg that provides an explicit description of the Poisson boundary of semisimple Lie groups. We will only state it in the special case of $G = \mathrm{SL}_d(\mathbb{R})$, $d \geq 2$.

THEOREM 3.18 (Furstenberg [**Fu62a**]). *Let $d \geq 2$ and $G \doteq \mathrm{SL}_d(\mathbb{R})$. Denote by $P < G$ the cocompact closed subgroup of upper triangular matrices. Then for every admissible measure $\mu \in \mathrm{Prob}(G)$, there exists a unique μ-stationary Borel probability measure $\nu \in \mathrm{Prob}(G/P)$ and moreover $(G/P, \nu)$ is the (G, μ)-Poisson boundary.*

In what follows, we will identify the function space $\mathrm{L}^\infty(B, \nu_B)$ of the (G, μ)-Poisson boundary (B, ν_B) with the space of bounded harmonic functions $\mathrm{Har}(G, \mu)$. We now investigate various qualitative and rigidity properties of the nonsingular action $G \curvearrowright (B, \nu_B)$.

COROLLARY 3.19. *The nonsingular action $G \curvearrowright (B, \nu_B)$ is ergodic.*

PROOF. Let $Y \subset B$ be any G-invariant measurable subset. Then

$$\Phi_\mu(\mathbf{1}_Y) = \nu_B(Y)\mathbf{1}_G = \Phi_\mu(\nu_B(Y)\mathbf{1}_B)$$

is a constant harmonic function. By injectivity of Φ_μ, we have $\mathbf{1}_Y = \nu_B(Y)\mathbf{1}_B$. This implies that $Y \subset B$ is null or conull. Thus, $G \curvearrowright (B, \nu_B)$ is ergodic. $\square$

We say that a (G, μ)-space (C, ν_C) is a (G, μ)-*boundary* if there exists a G-equivariant measurable factor map $\pi : (B, \nu_B) \to (C, \nu_C)$. We characterize (G, μ)-boundaries in the next result.

THEOREM 3.20. *Let (C, ν_C) be any (G, μ)-space. The following assertions are equivalent:*

(i) *(C, ν_C) is a (G, μ)-boundary.*
(ii) *For every compact model of $G \curvearrowright (C, \nu_C)$, the limit probability measures $(\nu_C)_\omega$ in Theorem 3.12 are Dirac masses for $\mathbf{P}$-almost every $\omega \in \Omega$.*
(iii) *There exists a compact model of $G \curvearrowright (C, \nu_C)$ such that the limit probability measures $(\nu_C)_\omega$ in Theorem 3.12 are Dirac masses for $\mathbf{P}$-almost every $\omega \in \Omega$.*

PROOF. (i) $\Rightarrow$ (ii) Abusing notation, we assume that (C, ν_C) is already a compact metrizable (G, μ)-boundary. Let $\pi : (B, \nu_B) \to (C, \nu_C)$ be any G-equivariant measurable factor map.

CLAIM 3.21. For $\mathbf{P}$-almost every $\omega \in \Omega$, the limit probability measure $(\nu_C)_\omega$ arising in Theorem 3.12 satisfies $(\nu_C)_\omega = \delta_{\pi(\overline{\omega})}$.

Indeed, the proof of Theorem 3.12 shows that for $\mathbf{P}$-almost every $\omega = (g_n)_{n \in \mathbb{N}} \in \Omega$, with respect to the weak*-topology, we have

$$\begin{aligned}(\nu_C)_\omega &= \lim_n g_{0*} \cdots g_{n*} \nu_C \\ &= \lim_n g_{0*} \cdots g_{n*} (\pi_* \nu_B) \\ &= \lim_n \pi_* g_{0*} \cdots g_{n*} \nu_B.\end{aligned}$$

Combining with the proof of Theorem 3.16, this further implies that for $\mathbf{P}$-almost every $\omega = (g_n)_{n \in \mathbb{N}} \in \Omega$ and every $f \in \mathrm{C}(C)$, we have

$$\begin{aligned}(\nu_C)_\omega(f) &= \lim_n (g_{0*} \cdots g_{n*} \nu_B)(f \circ \pi) \\ &= \lim_n \Phi_\mu(f \circ \pi)(g_0 \cdots g_n) \\ &= \widehat{\Phi_\mu(f \circ \pi)}(\omega) \\ &= (f \circ \pi)(\omega) \\ &= (f \circ \pi)(\overline{\omega}) = \delta_{\pi(\overline{\omega})}(f).\end{aligned}$$

This finishes the proof of Claim 3.21.

(ii) ⇒ (iii) This implication follows from Proposition 3.3.

(iii) ⇒ (i) Abusing notation, we assume that (C, ν_C) is already a compact metrizable (G, μ)-space for which the limit probability measures $(\nu_C)_\omega$ arising in Theorem 3.12 are Dirac masses for $\mathbf{P}$-almost every $\omega \in \Omega$. Define the G-equivariant measurable map $\pi : B \to C$ so that for $\mathbf{P}$-almost every $\omega \in \Omega$, we have $(\nu_C)_\omega = \delta_{\pi(\overline{\omega})}$. Then

$$\pi_* \nu_B = \int_\Omega \delta_{\pi(\overline{\omega})} \, \mathrm{d}\mathbf{P}(\omega) = \int_\Omega (\nu_C)_\omega \, \mathrm{d}\mathbf{P}(\omega) = \nu_C.$$

Then $\pi : (B, \nu_B) \to (C, \nu_C)$ is a G-equivariant measurable factor map and so (C, ν_C) is a (G, μ)-boundary. □

Let us point out that for any (G, μ)-boundary (C, ν_C), Claim 3.21 shows that there exists an essentially unique G-equivariant measurable factor map $\pi_C : (B, \nu_B) \to (C, \nu_C)$. Applying Theorem 3.20 to the case when $(C, \nu_C) = (B, \nu_B)$, for every compact model of $G \curvearrowright (B, \nu_B)$ and for $\mathbf{P}$-almost every $\omega \in \Omega$, we have

$$(\nu_B)_\omega = \lim_n g_{0*} \cdots g_{n*} \nu_B = \delta_{\overline{\omega}}. \tag{3.1}$$

In the next result, we show that the Poisson boundary behaves well with respect to factor groups. Let $N \lhd G$ be any normal closed subgroup and let $p : G \to G/N$ be the factor map. Denote by $\overline{\mu} = p_* \mu \in \mathrm{Prob}(G/N)$ and observe that $\overline{\mu} \in \mathrm{Prob}(G/N)$ is admissible. Using Proposition 3.3, denote by $(\overline{B}, \nu_{\overline{B}})$ the $(G/N, \overline{\mu})$-space that satisfies $\mathrm{L}^\infty(\overline{B}, \nu_{\overline{B}}) = \mathrm{L}^\infty(B, \nu_B)^N$.

Proposition 3.22. *Keep the same notation as above. Then $(\overline{B}, \nu_{\overline{B}})$ is the $(G/N, \overline{\mu})$-Poisson boundary.*

Proof. Denote by $\mathrm{Har}(G, \mu)^N \subset \mathrm{Har}(G, \mu)$ the G-invariant closed subspace of N-invariant bounded μ-harmonic functions. In view of Theorem 3.16, it suffices to prove that the well-defined G-equivariant unital positive linear contraction $\Psi : \mathrm{Har}(G/N, \overline{\mu}) \to \mathrm{Har}(G, \mu)^N : F \mapsto F \circ p$ is bijective. Indeed, Ψ is clearly injective. Next, let $F \in \mathrm{Har}(G, \mu)^N$. For every $h \in N$ and every $g \in G$, we have $F(g) = (\lambda(h)F)(g) = F(h^{-1}g) = F(g\, g^{-1}h^{-1}g)$. Thus we may define the bounded function $\overline{F} : G/N \to \mathbb{C}$ by the formula $\overline{F}(gN) = F(g)$ for every $g \in G$. Then $\overline{F} \in \mathrm{Har}(G/N, \overline{\mu})$ and $F = \overline{F} \circ p$. This shows that Ψ is surjective and finishes the proof. □

4. Furstenberg boundary map

The next fundamental result provides the existence and the uniqueness of Furstenberg boundary maps. As usual, we fix an admissible measure $\mu \in \mathrm{Prob}(G)$ and we denote by (B, ν_B) the (G, μ)-Poisson boundary. We follow the exposition given in [**BS04**, §2].

Theorem 3.23 (Furstenberg [**Fu62b**]). *Let (X, ν) be a compact metrizable (G, μ)-space. Then there exists an essentially unique G-equivariant* *measurable* boundary map $\beta_\nu : (B, \nu_B) \to \mathrm{Prob}(X) : b \mapsto \beta_\nu(b)$ *such that*

$$\nu = \int_B \beta_\nu(b)\, \mathrm{d}\nu_B(b).$$

Proof. By Theorem 3.12, there is a measurable map $\Omega \to \mathrm{Prob}(X) : \omega \mapsto \nu_\omega$ so that $g_{0*}\nu_{S(\omega)} = \nu_\omega$ for $\mathbf{P}$-almost every $\omega = (g_n)_{n\in\mathbb{N}} \in \Omega$ and $\nu = \int_\Omega \nu_\omega \, \mathrm{d}\mathbf{P}(\omega)$. Note that for every $g \in G$ and $\mathbf{P}$-almost every $\omega = (g_n)_{n\in\mathbb{N}} \in \Omega$, with respect to the weak*-topology, we have

$$\nu_{T(\omega)} = \lim_n (g_0 g_1)_* \cdots g_{n*}\nu = \lim_n g_{0*} g_{1*} \cdots g_{n*}\nu = \nu_\omega$$

and

$$g_*\nu_\omega = \lim_n g_* g_{0*} g_{1*} \cdots g_{n*}\nu = \lim_n (g g_0)_* g_{1*} \cdots g_{n*}\nu = \nu_{g\cdot\omega}.$$

These properties imply that the G-equivariant measurable map $\beta_\nu : B \to \mathrm{Prob}(X) : b \mapsto \beta_\nu(b)$ where $\beta_\nu(b) \doteq \nu_\omega$ with $b = \overline{\omega} \in B$ is well-defined. Moreover, we have

$$\nu = \int_\Omega \nu_\omega \, \mathrm{d}\mathbf{P}(\omega) = \int_B \beta_\nu(b)\, \mathrm{d}\nu_B(b).$$

This proves the existence of the boundary map $\beta_\nu : B \to \mathrm{Prob}(X)$.

Let now $\beta : B \to \mathrm{Prob}(X) : b \mapsto \beta(b)$ be any G-equivariant measurable map such that $\nu = \int_B \beta(b)\, \mathrm{d}\nu_B(b)$. Then the (G, μ)-space $(\mathrm{Prob}(X), \beta_*\nu_B)$ is a (G, μ)-boundary. Recall that the barycenter map $\mathrm{Bar} : \mathrm{Prob}(\mathrm{Prob}(X)) \to \mathrm{Prob}(X)$ is defined by the formula

$$\forall \psi \in \mathrm{Prob}(\mathrm{Prob}(X)), \quad \mathrm{Bar}(\psi) \doteq \int_{\mathrm{Prob}(X)} \eta \, \mathrm{d}\psi(\eta).$$

Since $G \curvearrowright \mathrm{Prob}(X)$ is weak*-continuous affine, the barycenter map $\mathrm{Bar} : \mathrm{Prob}(\mathrm{Prob}(X)) \to \mathrm{Prob}(X)$ is G-equivariant. By assumption, we have $\mathrm{Bar}(\beta_*\nu_B) = \int_B \beta(b)\, \mathrm{d}\nu_B = \nu$. Theorem 3.20 implies that for $\mathbf{P}$-almost every $\omega \in \Omega$, with respect to the weak*-topology, we have

$$\begin{aligned}
\beta(\overline{\omega}) = \mathrm{Bar}(\delta_{\beta(\overline{\omega})}) &= \mathrm{Bar}\left(\lim_n g_{0*} g_{1*} \cdots g_{n*}(\beta_*\nu_B)\right) \\
&= \lim_n \mathrm{Bar}\left(g_{0*} g_{1*} \cdots g_{n*}(\beta_*\nu_B)\right) \\
&= \lim_n g_{0*} g_{1*} \cdots g_{n*} \mathrm{Bar}(\beta_*\nu_B) \\
&= \lim_n g_{0*} g_{1*} \cdots g_{n*}\nu \\
&= \nu_\omega = \beta_\nu(\overline{\omega}).
\end{aligned}$$

This proves the uniqueness of the boundary map $\beta_\nu : B \to \mathrm{Prob}(X)$. □

Recall that for any (G, μ)-boundary (C, ν_C), there exists an essentially unique G-equivariant measurable factor map $\pi_C : (B, \nu_B) \to (C, \nu_C)$. We give the following functional analytic interpretation of the above result. As before, we may regard $\mathrm{L}^\infty(C) \subset \mathrm{L}^\infty(B)$ as a G-invariant von Neumann subalgebra such that $\nu_C = \nu_B|_{\mathrm{L}^\infty(C)}$ via the G-equivariant weak*-continuous unital $*$-homomorphism $\pi_C^* : \mathrm{L}^\infty(C) \to \mathrm{L}^\infty(B) : f \mapsto f \circ \pi_C$ that satisfies $\nu_B \circ \pi_C^* = \nu_C$.

COROLLARY 3.24. *Let $\Phi : \mathrm{L}^\infty(C) \to \mathrm{L}^\infty(B)$ be any G-equivariant weak*-continuous unital positive map such that $\nu_B \circ \Phi = \nu_C$. Then for every $f \in \mathrm{L}^\infty(C)$, we have $\Phi(f) = f$.*

PROOF. Using Proposition 3.3, we may assume that (C, ν_C) is a compact metrizable (G, μ)-space. Regard $\mathrm{C}(C) \subset \mathrm{L}^\infty(C)$ and consider the restriction $\Phi|_{\mathrm{C}(C)} : \mathrm{C}(C) \to \mathrm{L}^\infty(B)$. By duality, we obtain the G-equivariant measurable boundary map $\beta_{\nu_C} : B \to \mathrm{Prob}(C)$ such that $\nu_C = \int_B \beta_{\nu_C}(b) \, \mathrm{d}\nu_B(b)$. By Theorems 3.20 and 3.23 and Claim 3.21, we know that for $\mathbf{P}$-almost every $\omega \in \Omega$, we have $\beta_{\nu_C}(\overline{\omega}) = (\nu_C)_\omega = \delta_{\pi_C(\overline{\omega})}$. This implies that for every $f \in \mathrm{C}(C)$ and $\mathbf{P}$-almost every $\omega \in \Omega$, we have $\Phi(f)(\overline{\omega}) = \beta_{\nu_C}(\omega)(f) = \delta_{\pi_C(\overline{\omega})}(f) = f(\pi_C(\overline{\omega})) = f(\overline{\omega})$ and so $\Phi(f) = f$. Since Φ is weak*-continuous and since $\mathrm{C}(C) \subset \mathrm{L}^\infty(C)$ is weak*-dense, it follows that for every $f \in \mathrm{L}^\infty(C)$, we have $\Phi(f) = f$. $\square$

The next corollary shows that the limit probability measures from Theorem 3.12 behave well under equivariant measurable factor maps.

COROLLARY 3.25. *Let (X, ν) and (Y, η) be compact metrizable (G, μ)-spaces and $\pi : (X, \nu) \to (Y, \eta)$ any G-equivariant measurable factor map. Then for $\mathbf{P}$-almost every $\omega \in \Omega$, we have $\pi_* \nu_\omega = \eta_\omega$.*

PROOF. Up to modifying π on a ν-conull measurable subset, we may assume that $\pi : X \to Y$ is Borel. Denote by $\pi_* : \mathrm{Prob}(X) \to \mathrm{Prob}(Y)$ the corresponding Borel map. By [Zi84, Proposition B.5]), there exists a ν-conull G-invariant Borel subset $X_0 \subset X$ such that $\pi|_{X_0} : X_0 \to Y$ is strictly G-equivariant. By Theorem 3.23, there exists an essentially unique G-equivariant measurable boundary map $\beta_\nu : B \to \mathrm{Prob}(X) : \overline{\omega} \mapsto \nu_\omega$ (resp. $\beta_\eta : B \to \mathrm{Prob}(Y) : \overline{\omega} \mapsto \eta_\omega$) so that $\nu = \mathrm{Bar}(\beta_{\nu *} \nu_B)$ (resp. $\eta = \mathrm{Bar}(\beta_{\eta *} \nu_B)$). Since $\nu(X \setminus X_0) = 0$, Remark 3.13 implies that for $\mathbf{P}$-almost every $\omega \in \Omega$, we have $\nu_\omega(X \setminus X_0) = 0$. Then we may consider the G-equivariant measurable map $\pi_* \circ \beta_\nu : B \to \mathrm{Prob}(Y) : \omega \mapsto \pi_* \nu_\omega$. For every $f \in \mathrm{C}(Y)$, we have $f \circ \pi \in \mathrm{B}(X)$ and Remark 3.13 implies that

$$\begin{aligned}
\int_\Omega \eta_\omega(f) \, \mathrm{d}\mathbf{P}(\omega) = \eta(f) &= (\pi_* \nu)(f) = \nu(f \circ \pi) \\
&= \int_\Omega \nu_\omega(f \circ \pi) \, \mathrm{d}\mathbf{P}(\omega) \\
&= \int_\Omega (\pi_* \nu_\omega)(f) \, \mathrm{d}\mathbf{P}(\omega).
\end{aligned}$$

This implies that

$$\int_\Omega \eta_\omega \, \mathrm{d}\mathbf{P}(\omega) = \int_\Omega \pi_* \nu_\omega \, \mathrm{d}\mathbf{P}(\omega).$$

By uniqueness in Theorem 3.23, it follows that for $\mathbf{P}$-almost every $\omega \in \Omega$, we have $\pi_* \nu_\omega = \eta_\omega$. $\square$

The next corollary allows to identify conditional measures and limit measures.

Corollary 3.26. *Let (Y, η) be any compact metrizable (G, μ)-space, (C, ν_C) any (G, μ)-boundary and $\pi : (Y, \eta) \to (C, \nu_C)$ any relatively measure preserving G-equivariant measurable factor map. Denote by $\pi_C : (B, \nu_B) \to (C, \nu_C)$ the essentially unique G-equivariant measurable factor map. Regard $\mathrm{L}^\infty(C) \subset \mathrm{L}^\infty(Y)$ as a G-invariant von Neumann algebra such that $\nu_C = \eta|_{\mathrm{L}^\infty(C)}$.*

Then the unique conditional expectation $\mathrm{E} : \mathrm{L}^\infty(Y) \to \mathrm{L}^\infty(C)$ such that $\nu_C \circ \mathrm{E} = \eta$ satisfies that for $\mathbf{P}$-almost every $\omega \in \Omega$ and every $f \in \mathrm{C}(Y)$, we have $\mathrm{E}(f)(\pi_C(\overline{\omega})) = \eta_\omega(f)$.

Proof. Consider the restriction $\mathrm{E}|_{\mathrm{C}(Y)} : \mathrm{C}(Y) \to \mathrm{L}^\infty(C)$. By duality, we obtain the G-equivariant measurable map $\beta : C \to \mathrm{Prob}(Y)$ such that $\eta = \int_C \beta(c)\, \mathrm{d}\nu_C(c)$. Then $\beta \circ \pi_C : B \to \mathrm{Prob}(Y)$ is a G-equivariant measurable map such that $\eta = \int_B (\beta \circ \pi_C)(b)\, \nu_B(b)$. By uniqueness in Theorem 3.23, it follows that $\beta \circ \pi_C = \beta_\eta$. This implies that for $\mathbf{P}$-almost every $\omega \in \Omega$, we have $\beta(\pi_C(\overline{\omega})) = \eta_\omega$. By definition of β, this further implies that for $\mathbf{P}$-almost every $\omega \in \Omega$ and every $f \in \mathrm{C}(Y)$, we have $\mathrm{E}(f)(\pi_C(\overline{\omega})) = \beta(\pi_C(\overline{\omega}))(f) = \eta_\omega(f)$. $\square$

The next corollary provides a useful criterion to deduce equality between (G, μ)-boundaries.

Corollary 3.27. *For every $i \in \{1, 2\}$, let (C_i, ν_{C_i}) be any (G, μ)-boundary and denote by $\pi_{C_i} : (B, \nu_B) \to (C_i, \nu_{C_i})$ the essentially unique G-equivariant measurable factor map. Assume that there exists a G-equivariant measurable factor map $\pi : (C_1, \nu_{C_1}) \to (C_2, \nu_{C_2})$.*

If $\pi : (C_1, \nu_{C_1}) \to (C_2, \nu_{C_2})$ is relatively measure preserving, then $\pi : (C_1, \nu_{C_1}) \to (C_2, \nu_{C_2})$ is an isomorphism.

Proof. By essential uniqueness in Theorem 3.20, we necessarily have $\pi_{C_2} = \pi \circ \pi_{C_1}$. As before, regard $\mathrm{L}^\infty(C_2) \subset \mathrm{L}^\infty(C_1) \subset \mathrm{L}^\infty(B)$ as G-invariant von Neumann subalgebras. By assumption, the conditional expectation $\mathrm{E} : \mathrm{L}^\infty(C_1) \to \mathrm{L}^\infty(C_2)$ is G-equivariant. Since $\mathrm{L}^\infty(C_2) \subset \mathrm{L}^\infty(B)$, we may regard $\mathrm{E} : \mathrm{L}^\infty(C_1) \to \mathrm{L}^\infty(B)$ as a G-equivariant weak*-continuous unital positive map such that $\nu_B \circ \mathrm{E} = \nu_{C_1}$. By Corollary 3.24, we have $\mathrm{E}(f) = f$ for every $f \in \mathrm{L}^\infty(C_1)$. This implies that $\pi : (C_1, \nu_{C_1}) \to (C_2, \nu_{C_2})$ is an isomorphism. $\square$

As a straightforward consequence of Theorem 3.23 and Corollary 3.27, we obtain that any (G, μ)-boundary (C, ν_C) for which the measure $\nu_C \in \mathrm{Prob}(C)$ is G-invariant is necessarily trivial. In particular, we infer the following characterization of triviality of the Poisson boundary.

Corollary 3.28. *The following assertions are equivalent:*

(i) *The Poisson boundary (B, ν_B) is trivial.*

(ii) *For every compact metrizable (G, μ)-space (X, ν), the measure ν is G-invariant.*

Proof. (i) $\Rightarrow$ (ii) By Theorem 3.23, since $(B, \nu_B) \cong (\{*\}, \delta_{\{*\}})$ is trivial, for every compact metrizable (G, μ)-space (X, ν), the boundary map $\beta_\nu : B \to \mathrm{Prob}(X)$ is essentially constant and its unique essential value is equal to ν, which is necessarily G-invariant.

(ii) $\Rightarrow$ (i) We may assume that (B, ν_B) is already a compact metrizable (G, μ)-space. By assumption, the measure ν_B is G-invariant. Then the G-equivariant map $\pi_{\{*\}} : (B, \nu_B) \to (\{*\}, \delta_{\{*\}})$ is relatively measure preserving. Corollary 3.27 implies that $(B, \nu_B) \cong (\{*\}, \delta_{\{*\}})$ is trivial. $\square$

The next corollary provides a strengthening of the ergodicity property of the Poisson boundary obtained in Corollary 3.19.

Corollary 3.29. *Let $G \curvearrowright (X, \nu)$ be any ergodic pmp action. Then the nonsingular action $G \curvearrowright (B \times X, \nu_B \otimes \nu)$ is ergodic.*

Proof. We may assume that both (B, ν_B) and (X, ν) are compact metrizable (G, μ)-spaces. Then $(B \times X, \nu_B \otimes \nu)$ is a compact metrizable (G, μ)-space. Denote by $p_X : B \times X \to X$ and $p_B : B \times X \to B$ the canonical G-equivariant factor maps. Let $Z \subset B \times X$ be any G-invariant measurable subset such that $(\nu_B \otimes \nu)(Z) > 0$. Define $\eta = \frac{1}{(\nu_B \otimes \nu)(Z)} (\nu_B \otimes \nu)|_Z$. Then $(B \times X, \eta)$ is still a compact metrizable (G, μ)-space. Since $G \curvearrowright (B, \nu_B)$ is ergodic, Lemma 3.9 implies that $p_B : (B \times X, \eta) \to (B, \nu_B)$ is a G-equivariant measurable factor map. Likewise, since $G \curvearrowright (X, \nu)$ is ergodic, $p_X : (B \times X, \eta) \to (X, \nu)$ is a G-equivariant measurable factor map. Then Corollary 3.25 implies that for $\mathbf{P}$-almost every $\omega \in \Omega$, we have $p_{B*}\eta_\omega = (\nu_B)_\omega = \delta_{\overline{\omega}}$ and $p_{X*}\eta_\omega = \nu_\omega = \nu$ and so $\eta_\omega = \delta_{\overline{\omega}} \otimes \nu$. This implies that

$$\eta = \int_\Omega \eta_\omega \, \mathrm{d}\mathbf{P}(\omega) = \int_\Omega \delta_{\overline{\omega}} \otimes \nu \, \mathrm{d}\mathbf{P}(\omega) = \nu_B \otimes \nu.$$

This further implies that $(\nu_B \otimes \nu)(Z) = 1$ and so the nonsingular action $G \curvearrowright (B \times X, \nu_B \otimes \nu)$ is ergodic. $\square$

5. Amenability and the Poisson boundary

For every $p \in [1, +\infty]$, we simply denote by $\mathrm{L}^p(G) = \mathrm{L}^p(G, \mathcal{B}(G), m_G)$ and by $\lambda : G \curvearrowright \mathrm{L}^p(G)$ the left translation action. Let $G \curvearrowright (X, \nu)$ be any nonsingular action and denote by $\sigma : G \curvearrowright \mathrm{L}^\infty(X)$ the corresponding weak*-continuous action. Simply write $\mathrm{L}^\infty(G \times X) = \mathrm{L}^\infty(G \times X, m_G \otimes \nu)$. Denote by $\lambda \otimes \sigma : G \curvearrowright \mathrm{L}^\infty(G \times X)$ the weak*-continuous action arising from the diagonal nonsingular action $G \curvearrowright (G \times X, m_G \otimes \nu)$.

Definition 3.30. We say that a nonsingular action $G \curvearrowright (X, \nu)$ is *amenable* if there exists a unital positive linear contractive mapping $\Phi : \mathrm{L}^\infty(G \times X) \to \mathrm{L}^\infty(X)$ such that

- For every $f \in \mathrm{L}^\infty(X)$, we have $\Phi(\mathbf{1}_G \otimes f) = f$.

- For every $g \in G$ and every $F \in \mathrm{L}^\infty(G \times X)$, we have

$$\Phi((\lambda \otimes \sigma)(g)F) = \sigma(g)\Phi(F).$$

We simply say that $\Phi : \mathrm{L}^\infty(G \times X) \to \mathrm{L}^\infty(X)$ is a *G-equivariant projection.*

Recall that $\mathrm{P}(G) = \{\mu \in \mathrm{L}^1(G) \mid \mu \geq 0 \text{ and } \|\mu\|_1 = 1\}$. For every $\mu \in \mathrm{L}^1(G)$ and every $F \in \mathrm{L}^\infty(G \times X)$, we denote by $(\mu \otimes \mathrm{id}_X)(F) \in \mathrm{L}^\infty(X)$ the unique element that satisfies

$$\forall \psi \in \mathrm{L}^1(X, \nu), \quad \psi((\mu \otimes \mathrm{id}_X)(F)) = (\mu \otimes \psi)(F).$$

If $\mu \in \mathrm{P}(G)$, then $\mu \otimes \mathrm{id}_X : \mathrm{L}^\infty(G \times X) \to \mathrm{L}^\infty(X)$ is a unital positive linear contractive mapping. If $(\mu_i)_{i \in I}$ is a net in $\mathrm{L}^1(G)$ such that $\lim_i \|\mu_i\|_1 = 0$, then for every $F \in \mathrm{L}^\infty(G \times X)$, we have $(\mu_i \otimes \mathrm{id}_X)(F) \to 0$ with respect to the weak*-topology.

Proposition 3.31. *The following assertions hold:*

(i) *The nonsingular translation action $G \curvearrowright (G, m_G)$ is amenable.*
(ii) *If G is amenable, then every nonsingular action $G \curvearrowright (X, \nu)$ is amenable.*
(iii) *For every amenable nonsingular action $G \curvearrowright (X, \nu)$ and every lattice $\Gamma < G$, the nonsingular action $\Gamma \curvearrowright (X, \nu)$ is amenable.*

Proof. (i) Fix $\mu \in \mathrm{P}(G)$. Define the unital positive linear contractive mapping $\Psi = \mu \otimes \mathrm{id}_G : \mathrm{L}^\infty(G \times G) \to \mathrm{L}^\infty(G)$. Then the following properties hold:

- For every $f \in \mathrm{L}^\infty(G)$, we have $\Psi(\mathbf{1}_G \otimes f) = \mu(\mathbf{1}_G)\, f = f$.
- For every $g \in G$ and every $F \in \mathrm{L}^\infty(G \times G)$, we have

$$\Psi((\mathrm{id}_G \otimes \lambda)(g)F) = (\mu \otimes \lambda(g))(F) = \lambda(g)\Psi(F).$$

Next consider the nonsingular automorphism $\theta : G \times G \to G \times G : (h, k) \mapsto (kh, k)$ and define the unital positive linear contractive mapping $\Phi : \mathrm{L}^\infty(G \times G) \to \mathrm{L}^\infty(G)$ by the formula $\Phi(F) = \Psi(F \circ \theta)$. Then the following properties hold:

- For every $f \in \mathrm{L}^\infty(G)$, we have

$$\Phi(\mathbf{1}_G \otimes f) = \Psi((\mathbf{1}_G \otimes f) \circ \theta) = \Psi(\mathbf{1}_G \otimes f) = f.$$

- For every $g \in G$ and every $F \in \mathrm{L}^\infty(G \times G)$, we have

$$\begin{aligned}\Phi((\lambda \otimes \lambda)(g)F) &= \Psi(F \circ (g^{-1} \otimes g^{-1}) \circ \theta) \\ &= \Psi(F \circ \theta \circ (\mathrm{id}_G \otimes g^{-1})) \\ &= \Psi((\mathrm{id}_G \otimes \lambda)(g)(F \circ \theta)) \\ &= \lambda(g)\Psi(F \circ \theta) \\ &= \lambda(g)\Phi(F).\end{aligned}$$

Thus, $\Phi : \mathrm{L}^\infty(G \times G) \to \mathrm{L}^\infty(G)$ is a G-equivariant projection and so the nonsingular translation action $G \curvearrowright (G, m_G)$ is amenable.

(ii) Since G is amenable, there exists a net of elements $(\mu_i)_{i \in I}$ in $\mathrm{P}(G)$ such that $\|\lambda(g)\mu_i - \mu_i\|_1 \to 0$ uniformly on compact sets (see the proof of Theorem 2.20(iii) $\Rightarrow$ (i)). Choose a nonprincipal ultrafilter $\mathcal{U}$ on I. Define the unital positive linear contractive mapping $\Phi : \mathrm{L}^\infty(G \times X) \to \mathrm{L}^\infty(X)$ by the formula

$$\forall F \in \mathrm{L}^\infty(G \times X), \quad \Phi(F) = \lim_{i \to \mathcal{U}} (\mu_i \otimes \mathrm{id}_X)(F)$$

where the above limit is taken with respect to the weak*-topology in $\mathrm{L}^\infty(X)$.

- For every $f \in \mathrm{L}^\infty(X)$, we have
$$\Phi(\mathbf{1}_G \otimes f) = \lim_{i \to \mathcal{U}} (\mu_i \otimes \mathrm{id}_X)(\mathbf{1}_G \otimes f) = \lim_{i \to \mathcal{U}} \mu_i(\mathbf{1}_G)\, f = f.$$
- For every $g \in G$ and every $F \in \mathrm{L}^\infty(G \times X)$, we have
$$\begin{aligned}\Phi((\lambda \otimes \sigma)(g)F) &= \lim_{i \to \mathcal{U}} (\mu_i \otimes \mathrm{id}_X)((\lambda \otimes \sigma)(g))F) \\ &= \lim_{i \to \mathcal{U}} (\lambda(g^{-1})\mu_i \otimes \sigma(g))(F) \\ &= \lim_{i \to \mathcal{U}} (\mu_i \otimes \sigma(g))(F) \\ &= \sigma(g) \left(\lim_{i \to \mathcal{U}} (\mu_i \otimes \mathrm{id}_X)(F) \right) \\ &= \sigma(g)\Phi(F)\end{aligned}$$
where in the third line we used the fact that $\|\lambda(g^{-1})\mu_i - \mu_i\|_1 \to 0$.

Thus, $\Phi : \mathrm{L}^\infty(G \times X) \to \mathrm{L}^\infty(X)$ is a G-equivariant projection and so the nonsingular action $G \curvearrowright (X, \nu)$ is amenable.

(iii) Denote by $\Phi : \mathrm{L}^\infty(G \times X) \to \mathrm{L}^\infty(X)$ the G-equivariant projection witnessing amenability of the nonsingular action $G \curvearrowright (X, \nu)$. Choose a Borel fundamental domain $\mathcal{F} \subset G$ so that $G = \mathcal{F} \cdot \Gamma$. Then $\mathcal{F}^{-1} \subset G$ is a Borel fundamental domain for the left translation action $\Gamma \curvearrowright G$. We may assume that $m_G(\mathcal{F}^{-1}) = 1$ so that $\eta \doteq m_G|_{\mathcal{F}^{-1}} \in \mathrm{Prob}(\mathcal{F}^{-1})$. Then $\theta : (\Gamma \times \mathcal{F}^{-1}, m_\Gamma \otimes \eta) \to (G, m_G) : (\gamma, y) \mapsto \gamma y$ is a measure space isomorphism. Moreover, for all $\gamma, s \in \Gamma$ and all $y \in \mathcal{F}$, we have $\theta(\gamma s, y) = \gamma\theta(s, y)$. This implies that the canonical inclusion $\mathrm{L}^\infty(\Gamma \times X) \subset \mathrm{L}^\infty(\Gamma \times \mathcal{F}^{-1} \times X) \cong \mathrm{L}^\infty(G \times X)$ is Γ-equivariant. Thus $\Psi \doteq \Phi|_{\mathrm{L}^\infty(\Gamma \times X)} : \mathrm{L}^\infty(\Gamma \times X) \to \mathrm{L}^\infty(X)$ is a Γ-equivariant projection. This shows that the nonsingular action $\Gamma \curvearrowright (X, \nu)$ is amenable. $\square$

As usual, let $\mu \in \mathrm{Prob}(G)$ be any admissible measure. Denote by (B, ν_B) the (G, μ)-Poisson boundary. The following theorem and its corollary will be very useful in Section 5.

Theorem 3.32 (Zimmer [Zi76]). *The nonsingular action $G \curvearrowright (B, \nu_B)$ is amenable.*

Proof. Recall the construction of the Poisson boundary. We have $(\Omega, \mathcal{F}, \mathbf{P}) = (G^{\mathbb{N}}, \mathcal{B}(G)^{\otimes \mathbb{N}}, \mu^{\otimes \mathbb{N}})$ and the nonsingular action $G \curvearrowright (\Omega, \mathbf{P})$ is given by $g \cdot (g_n)_{n \in \mathbb{N}} = (gg_0, g_1, \dots)$ for every $g \in G$ and every $(g_n)_{n \in \mathbb{N}} \in \Omega$.

By Proposition 3.31(i) and since μ is admissible, it follows that the nonsingular action $G \curvearrowright (\Omega, \mathbf{P})$ is amenable. Denote by $\Phi : \mathrm{L}^\infty(G \times \Omega) \to \mathrm{L}^\infty(\Omega)$ a G-equivariant projection.

Next, the nonsingular transformation $T : (\Omega, \mathbf{P}) \to (\Omega, \mathbf{P})$ is given by $T((g_n)_{n\in\mathbb{N}}) = (g_0 g_1, \dots)$ for every $(g_n)_{n\in\mathbb{N}} \in \Omega$. Note that the nonsingular action $G \curvearrowright (\Omega, \mathbf{P})$ commutes with the nonsingular transformation $T : (\Omega, \mathbf{P}) \to (\Omega, \mathbf{P})$. By construction, we have a G-equivariant measurable factor map $(\Omega, \mathbf{P}) \to (B, \nu_B) : \omega \mapsto \overline{\omega}$ such that $\mathrm{L}^\infty(B, \nu_B) = \mathrm{L}^\infty(\Omega, \mathbf{P})^T$. Choose a nonprincipal ultrafilter $\mathcal{U} \in \beta(\mathbb{N}) \setminus \mathbb{N}$ and define the unital positive linear mapping $\mathrm{E} : \mathrm{L}^\infty(\Omega, \mathbf{P}) \to \mathrm{L}^\infty(\Omega, \mathbf{P})$ by the formula

$$\forall F \in \mathrm{L}^\infty(\Omega, \mathbf{P}), \quad \mathrm{E}(F) \doteq \lim_{n \to \mathcal{U}} \frac{1}{n+1} \sum_{k=0}^{n} F \circ T^k$$

where the above limit is taken with respect to the weak*-topology. Since the action $G \curvearrowright \mathrm{L}^\infty(\Omega)$ is weak*-continuous and commutes with $T : \Omega \to \Omega$, it follows that $\mathrm{E} : \mathrm{L}^\infty(\Omega, \mathbf{P}) \to \mathrm{L}^\infty(\Omega, \mathbf{P})^T$ is a G-equivariant projection. Define $\Psi : \mathrm{L}^\infty(G \times B) \to \mathrm{L}^\infty(B)$ by the formula $\Psi(F) = \mathrm{E}(\Phi(F))$ for every $F \in \mathrm{L}^\infty(G \times B) \subset \mathrm{L}^\infty(G \times \Omega)$. Then $\Psi : \mathrm{L}^\infty(G \times B) \to \mathrm{L}^\infty(B)$ is a G-equivariant projection and so the nonsingular action $G \curvearrowright (B, \nu_B)$ is amenable. $\square$

Corollary 3.33. *For every lattice $\Gamma < G$, the nonsingular action $\Gamma \curvearrowright (B, \nu_B)$ is amenable.*

Proof. The proof is simply a combinination of Proposition 3.31(iii) and Theorem 3.32. $\square$

We conclude this section by stating an important characterization of amenable groups involving the Poisson boundary.

Theorem 3.34 (Kaimanovich–Vershik [**KV82**], Rosenblatt [**Ro81**]). *Let G be any locally compact second countable group. Then G is amenable if and only if there exists an admissible measure $\mu \in \mathrm{Prob}(G)$ for which the Poisson boundary (B, ν_B) is trivial.*

Proof. Assume that there exists an admissible measure $\mu \in \mathrm{Prob}(G)$ for which the Poisson boundary $(B, \nu_B) = \{*\}$ is trivial. Then $\mathrm{L}^\infty(B, \nu_B) = \mathrm{L}^\infty(\{*\}) = \mathbb{C}$. Since $G \curvearrowright (B, \nu_B)$ is amenable by Theorem 3.32, there exists a G-equivariant projection $\mathfrak{m} : \mathrm{L}^\infty(G \times \{*\}) \to \mathbb{C}$. In other words, $\mathfrak{m} : \mathrm{L}^\infty(G) \to \mathbb{C}$ is a left invariant mean. By Theorem 2.20, G is amenable. For the converse implication, we refer the reader to [**KV82**] and [**Ro81**]. $\square$

CHAPTER 4

Reduced 1-cohomology and applications

We introduce 1-cohomology theory for unitary representations. We explain the relationship between reduced 1-cohomology and harmonic cocycles. We prove Shalom's characterization of property (T) in terms of reduced 1-cohomology. We show that induction is injective in both usual and reduced 1-cohomology.

1. 1-cohomology theory for unitary representations

DEFINITION 4.1. Let $\pi : G \to \mathcal{U}(\mathcal{H}_\pi)$ be any strongly continuous unitary representation. We say the a map $b : G \to \mathcal{H}_\pi$ is a 1-*cocycle* for π if b is continuous and satisfies the 1-cocycle relation

$$\forall g, h \in \mathcal{H}_\pi, \quad b(gh) = b(g) + \pi(g)b(h). \tag{4.1}$$

We denote by $\mathrm{Z}^1(G, \pi)$ the space of all 1-cocycles for π. We say that a map $b : G \to \mathcal{H}_\pi$ is a 1-*coboundary* for π if there exists $\xi \in \mathcal{H}_\pi$ such that $b(g) = \pi(g)\xi - \xi$ for every $g \in G$. We denote by $\mathrm{B}^1(G, \pi)$ the space of all 1-coboundaries for π.

It is clear from the definition that any 1-coboundary for π is a 1-cocycle for π and so $\mathrm{B}^1(G, \pi) \subset \mathrm{Z}^1(G, \pi)$. We denote by

$$\mathrm{H}^1(G, \pi) \doteq \mathrm{Z}^1(G, \pi) / \mathrm{B}^1(G, \pi)$$

the 1-cohomology space for π. In what follows, since we will be mainly interested in 1-cohomology theory, we will simply use the terminology *cocycle* (resp. *coboundary*, *cohomology*) instead of 1-cocycle (resp. 1-coboundary, 1-cohomology).

Observe that whenever $b : G \to \mathcal{H}_\pi$ is a map that satisfies the cocycle relation (4.1) and that is continuous at $e \in G$, then b is continuous on G and so $b \in \mathrm{Z}^1(G, \pi)$. The next result provides a useful criterion for a map $b : G \to \mathcal{H}_\pi$ to be a cocycle.

LEMMA 4.2. *Let $\pi : G \to \mathcal{U}(\mathcal{H}_\pi)$ be any strongly continuous unitary representation. Let $b : G \to \mathcal{H}_\pi$ be any map that satisfies the following conditions:*

- *The map b satisfies the cocycle relation (4.1).*
- *The function $G \to \mathbb{C} : g \mapsto \|b(g)\|$ is measurable.*
- *The subset $b(G) \subset \mathcal{H}_\pi$ is separable.*

Then $b \in \mathrm{Z}^1(G, \pi)$.

PROOF. The proof is somewhat similar to the one of Lemma 2.2. It suffices to show that the map $b : G \to \mathcal{H}_\pi : g \mapsto b(g)$ is continuous at $e \in G$. Let $Q \subset G$ be any symmetric compact neighborhood of $e \in G$. Consider the compactly generated open subgroup $H \doteq \bigcup_{n \geq 1} Q^n < G$. It further suffices to show that the map $b|_H : H \to \mathcal{H}_\pi : g \mapsto b(g)$ is continuous at $e \in H$. Thus, we may as well assume that G is σ-compact.

As usual, we denote by m_G a left invariant Haar measure on G. Fix $\varepsilon > 0$ and define the measurable subset $B \doteq \{g \in G \mid \|b(g)\| < \varepsilon/2\}$. By the cocycle relation (4.1), we have $B^{-1} = B$ and $B^2 \subset \{g \in G \mid \|b(g)\| < \varepsilon\}$. Take a sequence $(g_n)_{n \in \mathbb{N}}$ in G such that $\{b(g_n) \mid n \in \mathbb{N}\}$ is dense in $b(G)$. For every $g \in G$, there exists $n \in \mathbb{N}$ such that $\|b(g) - b(g_n)\| < \varepsilon/2$ and so

$$\|b(g_n^{-1} g)\| = \|\pi(g_n) b(g_n^{-1} g)\| = \|b(g) - b(g_n)\| < \varepsilon/2.$$

This implies that $G = \bigcup_{n \in \mathbb{N}} g_n B$ and hence $m_G(B) > 0$. Arguing as in the proof of Lemma 2.2, this further implies that B^2 contains an open neighborhood of $e \in G$ and so does $\{g \in G \mid \|b(g)\| < \varepsilon\}$. This implies that $b \in \mathrm{Z}^1(G, \pi)$. □

Observe that when G is σ-compact and $b \in \mathrm{Z}^1(G, \pi)$, the subset $b(G) \subset \mathcal{H}_\pi$ is separable. Indeed, write $G = \bigcup_{n \in \mathbb{N}} Q_n$ with $Q_n \subset G$ a compact subset for every $n \in \mathbb{N}$. Then $b(G) = \bigcup_{n \in \mathbb{N}} b(Q_n)$. Since $b : G \to \mathcal{H}_\pi$ is continuous, $b(Q_n)$ is a compact subset of the metric space $\mathcal{H}_\pi$ and so $b(Q_n) \subset \mathcal{H}_\pi$ is separable. This implies that $b(G) \subset \mathcal{H}_\pi$ is separable.

We will often use the following elementary result.

LEMMA 4.3. *Let* $\pi : G \to \mathcal{U}(\mathcal{H}_\pi)$ *be any strongly continuous unitary representation and* $b \in \mathrm{Z}^1(G, \pi)$ *any cocycle. If* b *is bounded, then* $b \in \mathrm{B}^1(G, \pi)$.

PROOF. Since the subset $b(G) \subset \mathcal{H}_\pi$ is bounded, we may consider its circumcenter $\xi \in \mathcal{H}_\pi$. By uniqueness of the circumcenter, using G-invariance and the cocycle relation (4.1), we obtain $\xi = b(g) + \pi(g)\xi$ for every $g \in G$. Thus, we have $b \in \mathrm{B}^1(G, \pi)$. □

There is a useful geometric interpretation of cocycles that goes as follows. Regard $\mathcal{H}_\pi$ as an *affine* Hilbert space and denote by $\mathrm{Aff}(\mathcal{H}_\pi)$ the group of affine transformations. Define the continuous map $\alpha_{\pi,b} : G \to \mathrm{Aff}(\mathcal{H}_\pi)$ by the formula $\alpha_{\pi,b}(g)(\xi) = \pi(g)\xi + b(g)$. The cocycle relation (4.1) implies that $\alpha_{\pi,b} : G \to \mathrm{Aff}(\mathcal{H}_\pi)$ is a group homomorphism and hence defines an affine isometric action of G on $\mathcal{H}_\pi$. Then π corresponds to the *linear* part of $\alpha_{\pi,b}$ and b corresponds to the *translation* part of $\alpha_{\pi,b}$. It is straightforward to see that b is a coboundary if and only if $\alpha_{\pi,b}$ admits a fixed point.

Endowed with the topology of uniform convergence on compact subsets of G, the space $\mathrm{Z}^1(G, \pi)$ is a locally convex topological vector space.

- Assume moreover that G is *σ-compact* and choose an increasing sequence of compact subsets $Q_n \subset G$ such that $G = \bigcup_{n\in\mathbb{N}} Q_n$. The family of seminorms $p_{Q_n} : \mathrm{Z}^1(G,\pi) \to \mathbb{R}_+$ defined by $p_{Q_n}(b) \doteq \sup\{\|b(g)\| \mid g \in Q_n\}$ is separating. Moreover, the metric d defined on $\mathrm{Z}^1(G,\pi)$ by the formula
$$\forall b,c \in \mathrm{Z}^1(G,\pi), \quad \mathsf{d}(b,c) \doteq \sum_{n\in\mathbb{N}} \frac{1}{2^{n+1}} \frac{p_{Q_n}(b-c)}{1+p_{Q_n}(b-c)}$$
is complete. Thus, $\mathrm{Z}^1(G,\pi)$ is a Fréchet space.
- Assume moreover that G is *compactly generated* and choose a compact subset $Q \subset G$ such that $e \in Q$ and $G = \bigcup_{n\geq 1} Q^n$. Then p_Q is a complete norm on $\mathrm{Z}^1(G,\pi)$ and so $\mathrm{Z}^1(G,\pi)$ is a Banach space.

The subspace $\mathrm{B}^1(G,\pi) \subset \mathrm{Z}^1(G,\pi)$ need not be closed. In that respect, we introduce the *reduced cohomology* space for π by
$$\overline{\mathrm{H}}^1(G,\pi) \doteq \mathrm{Z}^1(G,\pi)/\overline{\mathrm{B}^1(G,\pi)}.$$
The following result clarifies when $\mathrm{B}^1(G,\pi) \subset \mathrm{Z}^1(G,\pi)$ is closed.

Proposition 4.4 (Guichardet [**Gu72**]). *Let $\pi : G \to \mathcal{U}(\mathcal{H}_\pi)$ be any strongly continuous unitary representation.*

(i) *If π does not have almost invariant vectors, then $\mathrm{B}^1(G,\pi)$ is closed in $\mathrm{Z}^1(G,\pi)$. In that case, we have $\mathrm{H}^1(G,\pi) = \overline{\mathrm{H}}^1(G,\pi)$.*

(ii) *Assume that G is σ-compact and that π is ergodic. If $\mathrm{B}^1(G,\pi)$ is closed in $\mathrm{Z}^1(G,\pi)$, then π does not have almost invariant vectors.*

Proof. Define the continuous linear mapping
$$\partial : \mathcal{H}_\pi \to \mathrm{B}^1(G,\pi) : \xi \mapsto (g \mapsto \pi(g)\xi - \xi).$$
(i) Since π does not have almost invariant vectors, there exists a compact subset $Q \subset G$ and $\varepsilon > 0$ such that
$$\forall \xi \in \mathcal{H}_\pi, \quad p_Q(\partial\xi) \geq \varepsilon\|\xi\|. \tag{4.2}$$
Let $b \in \mathrm{Z}^1(G,\pi)$ and $(\xi_i)_{i\in I}$ be a net in $\mathcal{H}_\pi$ such that $\partial\xi_i \to b$ in $\mathrm{Z}^1(G,\pi)$. There exists $i_0 \in I$ such that for every $i \geq i_0$, we have $p_Q(\partial\xi_i - b) \leq 1$. Then (4.2) implies that
$$\forall i \geq i_0, \quad \|\xi_i\| \leq \frac{1}{\varepsilon}(p_Q(\partial\xi_i - b) + p_Q(b)) \leq \frac{1}{\varepsilon}(1 + p_Q(b)) \doteq \kappa. \tag{4.3}$$
For every compact subset $C \subset G$, we may choose $i_C \geq i_0$ so that $p_C(\partial\xi_{i_C} - b) \leq 1$. Combining with (4.3), we obtain
$$p_C(b) \leq p_C(b - \partial\xi_{i_C}) + p_C(\partial\xi_{i_C}) \leq 1 + 2\kappa.$$
Since b is bounded on G, it follows that $b \in \mathrm{B}^1(G,\pi)$ by Lemma 4.3. This shows that $\mathrm{B}^1(G,\pi) \subset \mathrm{Z}^1(G,\pi)$ is closed.

(ii) Since π is ergodic, the linear mapping $\partial : \mathcal{H}_\pi \to \mathrm{B}^1(G,\pi)$ is bijective. Since G is σ-compact and since $\mathrm{B}^1(G,\pi) \subset \mathrm{Z}^1(G,\pi)$ is closed, $\mathrm{B}^1(G,\pi)$ is a

Fréchet space. By the open mapping theorem (see [**Ru91**, Corollaries 2.12]), the inverse linear mapping $\partial^{-1} : \mathrm{B}^1(G,\pi) \to \mathcal{H}_\pi$ is continuous. Assume by contradiction that π has almost invariant vectors. Choose an increasing sequence of compact subsets $Q_n \subset G$ such that $G = \bigcup_{n\in\mathbb{N}} Q_n$. Then for every $n \in \mathbb{N}$, there exists a unit vector $\xi_n \in \mathcal{H}_\pi$ such that $p_{Q_n}(\partial\xi_n) \leq 2^{-n}$. This implies that $\mathsf{d}(\partial\xi_n, 0) \leq 2^{-(n-1)} \to 0$ (where d is the complete metric defined using $(Q_n)_{n\in\mathbb{N}}$). Thus, we have $\xi_n = \partial^{-1}(\partial\xi_n) \to 0$, which is a contradiction. □

In the next result, we obtain a characterization of property (T) in terms of 1-cohomology theory.

THEOREM 4.5 (Delorme–Guichardet [**De76, Gu72**]). *The following assertions hold:*

(i) *If G has property* (T)*, then for every strongly continuous unitary representation π, we have $\mathrm{H}^1(G,\pi) = 0$.*
(ii) *Assume that G is σ-compact. If $\mathrm{H}^1(G,\pi) = 0$ for every strongly continuous unitary representation π, then G has property* (T)*.*

PROOF. (i) Let $\pi : G \to \mathcal{U}(\mathcal{H}_\pi)$ be any strongly continuous unitary representation and $b \in \mathrm{Z}^1(G,\pi)$ any cocycle such that $b \notin \mathrm{B}^1(G,\pi)$. We will show that G does not have property (T). Then b is not bounded on G (see Lemma 4.3). Set $\mathcal{H}_\pi^{\otimes 0} = \mathbb{C}\Omega$ with $\|\Omega\| = 1$ and

$$\mathsf{exp}(\mathcal{H}_\pi) \doteq \bigoplus_{n\in\mathbb{N}} \mathcal{H}_\pi^{\otimes n} \quad \text{and} \quad \forall \xi \in \mathcal{H}_\pi,\ \mathsf{exp}(\xi) \doteq \sum_{n\in\mathbb{N}} \frac{1}{\sqrt{n!}} \xi^{\otimes n} \in \mathsf{exp}(\mathcal{H}_\pi).$$

Note that $\Omega = \mathsf{exp}(0)$. Denote by $\mathcal{K}$ the closure in $\mathsf{exp}(\mathcal{H}_\pi)$ of the linear span of $\{\mathsf{exp}(\xi) \mid \xi \in \mathcal{H}_\pi\}$. For every $t > 0$, define the strongly continuous unitary representation $\rho_t : G \to \mathcal{U}(\mathcal{K})$ by the formula

$$\rho_t(g)\mathsf{exp}(\xi) \doteq \exp(-\frac{t^2}{2}\|b(g)\|^2 - \Re\langle \pi(g)\xi, tb(g)\rangle)\,\mathsf{exp}(\pi(g)\xi + tb(g)) \tag{4.4}$$

for every $g \in G$ and every $\xi \in \mathcal{H}_\pi$. One easily checks that for every $g \in G$, the mapping $\rho_t(g) : \mathrm{Vect}\,\{\mathsf{exp}(\xi) \mid \xi \in \mathcal{H}_\pi\} \to \mathcal{K}$ extends to a well-defined unitary operator $\rho_t(g) \in \mathcal{U}(\mathcal{K})$. Moreover, $\rho : G \to \mathcal{U}(\mathcal{K})$ is a group homomorphism using the cocycle relation (4.1). Finally, $\rho : G \to \mathcal{U}(\mathcal{K})$ is strongly continuous by (4.4) and by density of $\mathrm{Vect}\,\{\mathsf{exp}(\xi) \mid \xi \in \mathcal{H}_\pi\}$ in $\mathcal{K}$. Then for every $g \in G$, we have

$$\exp(-\frac{t^2}{2}\|b(g)\|^2) = \langle \rho_t(g)\Omega, \Omega\rangle. \tag{4.5}$$

Denote by $\mathcal{K}_t$ the closure in $\mathcal{K}$ of the linear span of $\rho_t(G)\Omega$. We still denote by $\rho_t : G \to \mathcal{U}(\mathcal{K}_t)$ the strongly continuous unitary representation defined on the $\rho_t(G)$-invariant subspace $\mathcal{K}_t \subset \mathcal{K}$. Since b is not bounded on G, there exists a sequence $(g_n)_{n\in\mathbb{N}}$ in G such that $\lim_n \|b(g_n)\| = +\infty$. Then (4.1) and (4.5) imply that for every $g \in G$, we have

$$\langle \rho_t(g_n)\Omega, \rho_t(g)\Omega\rangle = \langle \rho_t(g^{-1}g_n)\Omega, \Omega\rangle$$

$$= \exp(-\frac{t^2}{2}\|\pi(g)b(g^{-1}g_n)\|^2)$$
$$= \exp(-\frac{t^2}{2}\|b(g) - b(g_n)\|^2) \to 0.$$

This implies that $\rho_t(g_n)\Omega \to 0$ weakly in $\mathcal{K}_t$. This further implies that ρ_t is ergodic. Indeed, let $\zeta \in (\mathcal{K}_t)^G$ be any $\rho_t(G)$-invariant vector. Then for every $g \in G$, we have

$$\langle \zeta, \rho_t(g)\Omega \rangle = \langle \rho_t(g_n g^{-1})\zeta, \rho_t(g_n g^{-1})\rho_t(g)\Omega \rangle = \langle \zeta, \rho_t(g_n)\Omega \rangle \to 0.$$

It follows that $\zeta = 0$ and so ρ_t is ergodic.

Define $\mathcal{K}_\rho \doteq \bigoplus_{n\geq 1} \mathcal{K}_{1/n}$ and $\rho \doteq \bigoplus_{n\geq 1} \rho_{1/n}$. Note that ρ is still ergodic. Define $\Omega_n \in \mathcal{K}_\rho$ the unit vector whose n-th component is equal to $\Omega \in \mathcal{K}_{1/n}$. Then (4.5) implies that $(\Omega_n)_{n\geq 1}$ is a sequence of almost invariant vectors for ρ and so $1_G \prec \rho$. This implies that G does not have property (T).

(ii) If G does not have property (T), then there exists an ergodic strongly continuous unitary representation π that has almost invariant vectors. Then Proposition 4.4(ii) implies that $\mathrm{B}^1(G,\pi) \neq \overline{\mathrm{B}^1(G,\pi)}$. In particular, we have $\mathrm{B}^1(G,\pi) \neq \mathrm{Z}^1(G,\pi)$ and so $\mathrm{H}^1(G,\pi) \neq 0$. □

2. Reduced cohomology and harmonic cocycles

From now on and for the rest of this chapter, we will assume that the locally compact second countable group G is compactly generated. More precisely, we use the following setup.

TERMINOLOGY 4.6. Let G be any compactly generated lcsc group. Fix a symmetric compact neighborhood $Q \subset G$ of $e \in G$ such that $G = \bigcup_{n\geq 1} Q^n$. Define the word length $\ell_Q : G \to \mathbb{N}$ on G associated with Q by the formula

$$\forall g \in G, \quad \ell_Q(g) \doteq \min\{n \in \mathbb{N} \mid g \in Q^n\}.$$

We say that a symmetric Borel probability measure $\mu \in \mathrm{Prob}(G)$ is *cohomologically adapted* if

(i) μ is absolutely continuous with respect to the Haar measure on G.
(ii) The support $\mathrm{supp}(\mu)$ generates G as a semigroup, that is, $G = \bigcup_{n\geq 1} \mathrm{supp}(\mu)^n$.
(iii) μ has a finite second moment, that is, $\int_G \ell_Q(g)^2 \,\mathrm{d}\mu(g) < +\infty$.

Let $\pi : G \to \mathcal{U}(\mathcal{H}_\pi)$ be any strongly continuous unitary representation. We do not assume that $\mathcal{H}_\pi$ is separable. Let $b \in \mathrm{Z}^1(G,\pi)$ be any cocycle. Then for every $g \in G$, we have $\|b(g)\| \leq \ell_Q(g)\, p_Q(b)$. It follows that the continuous function $G \to \mathbb{R}_+ : g \mapsto \|b(g)\|$ belongs to $\mathrm{L}^2(G,\mu)$ and hence we may consider the element $b(\mu) \in \mathcal{H}_\pi$ defined by the formula

$$\forall \xi \in \mathcal{H}_\pi, \quad \langle b(\mu), \xi \rangle = \int_G \langle b(h), \xi \rangle \,\mathrm{d}\mu(h).$$

We will simply write $b(\mu) = \int_G b(h)\,\mathrm{d}\mu(h)$. We say that b is *μ-harmonic* if $b(\mu) = 0$. Using the cocycle relation (4.1), b is μ-harmonic if and only if

$$\forall g \in G, \quad b(g) = \int_G b(gh)\,\mathrm{d}\mu(h).$$

Denote by $\mathrm{Har}_\mu(G,\pi) \subset \mathrm{Z}^1(G,\pi)$ the closed subspace of all μ-harmonic cocycles.

Recall that $(\mathrm{Z}^1(G,\pi), p_Q)$ is a Banach space. We may also endow the space $\mathrm{Z}^1(G,\pi)$ with the sesquilinear form $\langle\,\cdot\,,\cdot\,\rangle_\mu$ defined by the formula

$$\forall b, c \in \mathrm{Z}^1(G,\pi), \quad \langle b, c\rangle_\mu \doteq \int_G \langle b(h), c(h)\rangle\,\mathrm{d}\mu(h).$$

Since $G = \bigcup_{n\geq 1} \mathrm{supp}(\mu)^n$, it is easy to see that $\|\cdot\|_\mu$ is a norm on $\mathrm{Z}^1(G,\pi)$. More generally, we prove the following useful result.

THEOREM 4.7. *The space $\mathrm{Z}^1(G,\pi)$ endowed with the sesquilinear form $\langle\,\cdot\,,\cdot\,\rangle_\mu$ is a Hilbert space. For every compact subset $K \subset G$ such that $Q \subset K$, there exists a constant $\kappa > 0$ such that*

$$\frac{1}{\kappa}\,p_K \leq \|\cdot\|_\mu \leq \kappa\,p_K.$$

Moreover, we have the following orthogonal decomposition

$$\mathrm{Z}^1(G,\pi) = \overline{\mathrm{B}^1(G,\pi)} \oplus \mathrm{Har}_\mu(G,\pi). \tag{4.6}$$

We may and will identify $\overline{\mathrm{H}}^1(G,\pi) \cong \mathrm{Har}_\mu(G,\pi)$.

PROOF. Firstly, we prove that $\mathrm{Z}^1(G,\pi)$ is a Hilbert space. Let $(b_n)_n$ be any $\|\cdot\|_\mu$-Cauchy sequence in $\mathrm{Z}^1(G,\pi)$. We want to show that $(b_n)_n$ admits a limit in $\mathrm{Z}^1(G,\pi)$ with respect to $\|\cdot\|_\mu$. Up to taking a subsequence, we may assume that $\|b_{n+1} - b_n\|_\mu \leq 2^{-(n+1)}$ for every $n \in \mathbb{N}$. Using Cauchy–Schwarz inequality, we have

$$\begin{aligned}
\int_G \sum_{n\in\mathbb{N}} \|b_{n+1}(h) - b_n(h)\|\,\mathrm{d}\mu(h) &= \sum_{n\in\mathbb{N}} \int_G \|b_{n+1}(h) - b_n(h)\|\,\mathrm{d}\mu(h) \\
&\leq \sum_{n\in\mathbb{N}} \|b_{n+1} - b_n\|_\mu \\
&\leq 1.
\end{aligned}$$

Since $\mathcal{H}_\pi$ is complete, it follows that $\lim_n b_n(g)$ exists in $\mathcal{H}_\pi$ for μ-almost every $g \in G$. Observe that for every $c \in \mathrm{Z}^1(G,\pi)$, we have

$$\begin{aligned}
\|c\|^2_{\mu*\mu} &= \int_{G\times G} \|c(gh)\|^2\,\mathrm{d}\mu(g)\mathrm{d}\mu(h) \\
&= \int_{G\times G} \|c(g) + \pi(g)c(h)\|^2\,\mathrm{d}\mu(g)\mathrm{d}\mu(h) \\
&\leq 2\|c\|^2_\mu + 2\|c\|^2_\mu = 4\|c\|^2_\mu.
\end{aligned}$$

More generally, for every $k \geq 1$ and every $c \in \mathrm{Z}^1(G, \pi)$, we have

$$\|c\|^2_{\mu^{*k}} \leq k^2 \|c\|^2_\mu.$$

The exact same reasoning as before shows that for every $k \geq 1$, $\lim_n b_n(g)$ exists in $\mathcal{H}_\pi$ for μ^{*k}-almost every $g \in G$.

Next, set

$$\Lambda \doteq \left\{ g \in G \mid \lim_n b_n(g) \text{ exists in } \mathcal{H}_\pi \right\}.$$

Claim 4.8. We have $\Lambda = G$.

Indeed, the previous reasoning shows that the Borel subset $\Lambda \subset G$ is not empty and $\mu^{*k}(G \setminus \Lambda) = 0$ for every $k \geq 1$. Moreover, the cocycle relation (4.1) implies that $\Lambda \subset G$ is a subgroup. By contradiction, assume that $\Lambda \neq G$. Then there exists $g \in G$ such that $g\Lambda \subset G \setminus \Lambda$. Since for every $k \geq 1$, we have $\mathrm{supp}(\mu)^k \subset \mathrm{supp}(\mu^{*k})$ and since $G = \bigcup_{k \geq 1} \mathrm{supp}(\mu)^k$, there exists $\ell \geq 1$ such that $g \in \mathrm{supp}(\mu^{*\ell})$. Since μ is absolutely continuous with respect to m_G, we may consider $f \doteq \frac{\mathrm{d}\mu}{\mathrm{d}m_G} \in \mathrm{L}^1(G, m_G)$ with $f \geq 0$ and $\|f\|_1 = 1$. Since the map $G \to \mathrm{L}^1(G) : h \mapsto \lambda(h)f$ is $\|\cdot\|_1$-continuous and since the measurable map $G \to \mathbb{R}_+ : x \mapsto \mu(x^{-1}g\Lambda)$ is bounded, Lebesgue's dominated convergence theorem implies that the map $G \to \mathbb{R}_+ : h \mapsto \mu^{*2}(h^{-1}g\Lambda)$ is continuous because

$$\begin{aligned}
\forall h \in G, \quad \mu^{*2}(h^{-1}g\Lambda) &= (\mu * \mu)(h^{-1}g\Lambda) \\
&= \int_G \mu(x^{-1}h^{-1}g\Lambda)\,\mathrm{d}\mu(x) \\
&= \int_G \mu(x^{-1}h^{-1}g\Lambda) f(x)\,\mathrm{d}m_G(x) \\
&= \int_G \mu(x^{-1}g\Lambda) f(h^{-1}x)\,\mathrm{d}m_G(x) \\
&= \int_G \mu(x^{-1}g\Lambda)(\lambda(h)f)(x)\,\mathrm{d}m_G(x).
\end{aligned}$$

Since $\mu^{*(2+\ell)}(G \setminus \Lambda) = 0$ and $g\Lambda \subset G \setminus \Lambda$, we have $\mu^{*(2+\ell)}(g\Lambda) = 0$. This further implies that

$$\int_G \mu^{*2}(h^{-1}g\Lambda)\,\mathrm{d}\mu^{*\ell}(h) = \mu^{*(2+\ell)}(g\Lambda) = 0.$$

Then for $\mu^{*\ell}$-almost every $h \in G$, we have $\mu^{*2}(h^{-1}g\Lambda) = 0$. By continuity, we infer that $\mu^{*2}(h^{-1}g\Lambda) = 0$ for every $h \in \mathrm{supp}(\mu^{*\ell})$. In particular, for $h = g \in \mathrm{supp}(\mu^{*\ell})$, we infer that $\mu^{*2}(g^{-1}g\Lambda) = \mu^{*2}(\Lambda) = 0$. Since $\mu^{*2}(G \setminus \Lambda) = 0$, we obtain $\mu^{*2}(G) = \mu(\Lambda) + \mu(G \setminus \Lambda) = 0$, which is absurd. Therefore, we have $\Lambda = G$, which means that $\lim_n b_n(g)$ exists in $\mathcal{H}_\pi$ for *every* $g \in G$. This finishes the proof of Claim 4.8.

Set $b(g) = \lim_n b_n(g) \in \mathcal{H}_\pi$ for every $g \in G$. We now prove that $b \in \mathrm{Z}^1(G, \pi)$. It is clear that the map $b : G \to \mathcal{H}_\pi$ satisfies the cocycle relation

(4.1). The function $G \to \mathbb{R}_+ : g \mapsto \|b(g)\|$ is measurable as pointwise limit of the continuous functions $G \to \mathbb{R}_+ : g \mapsto \|b_n(g)\|$. Since for every $n \in \mathbb{N}$, $b_n(G)$ is separable and since $b(G) \subset \overline{\bigcup_{n\in\mathbb{N}} b_n(G)}$, it follows that $b(G)$ is separable. Then Lemma 4.2 implies that $b \in \mathrm{Z}^1(G, \pi)$.

For every $p \geq n$, we have

$$\|b_p - b_n\|_\mu \leq \sum_{k=n}^{p-1} \|b_{k+1} - b_k\|_\mu \leq \sum_{k=n}^{p-1} 2^{-(k+1)} \leq 2^{-n}.$$

Fatou's lemma further implies that

$$\begin{aligned}
\|b - b_n\|_\mu^2 &= \int_G \|b(g) - b_n(g)\|^2 \, \mathrm{d}\mu(g) \\
&= \int_G \liminf_p \|b_p(g) - b_n(g)\|^2 \, \mathrm{d}\mu(g) \\
&\leq \liminf_p \int_G \|b_p(g) - b_n(g)\|^2 \, \mathrm{d}\mu(g) \\
&= \liminf_p \|b_p - b_n\|_\mu^2 \\
&\leq (2^{-n})^2.
\end{aligned}$$

Therefore $\lim_n \|b - b_n\|_\mu = 0$. This shows that the norm $\|\cdot\|_\mu$ is complete on $\mathrm{Z}^1(G, \pi)$ and so $(\mathrm{Z}^1(G, \pi), \langle \cdot , \cdot \rangle_\mu)$ is indeed a Hilbert space.

Let now $K \subset G$ be any compact subset such that $Q \subset K$. Then we have $G = \bigcup_{n\geq 1} K^n$ and hence p_K is a complete norm on $\mathrm{Z}^1(G, \pi)$. Since $Q \subset K$, we have $p_Q \leq p_K$. Observe that for every $b \in \mathrm{Z}^1(G, \pi)$, we have

$$\|b\|_\mu^2 = \int_G \|b(h)\|^2 \, \mathrm{d}\mu(h) \leq \int_G \ell_Q(h)^2 \, \mathrm{d}\mu(h) \cdot p_Q(b) \leq \int_G \ell_Q(h)^2 \, \mathrm{d}\mu(h) \cdot p_K(b).$$

In particular, we have $\|\cdot\|_\mu \leq \kappa_1 \, p_K$ where $\kappa_1 = (\int_G \ell_Q(h)^2 \, \mathrm{d}\mu(h))^{1/2}$. This further implies that the identity linear mapping

$$\iota : (\mathrm{Z}^1(G, \pi), p_K) \to (\mathrm{Z}^1(G, \pi), \|\cdot\|_\mu) : b \mapsto b$$

is continuous and bijective. Since both $(\mathrm{Z}^1(G, \pi), p_K)$ are $(\mathrm{Z}^1(G, \pi), \|\cdot\|_\mu)$ are Banach spaces, the open mapping theorem (see [**Ru91**, Corollaries 2.12]) implies that ι^{-1} is continuous. This further implies that there exists a constant $\kappa_2 > 0$ such that $p_K \leq k_2 \, \|\cdot\|_\mu$.

Secondly, we prove the orthogonal decomposition (4.6). Indeed,

$$\begin{aligned}
b \in (\mathrm{B}^1(G, \pi))^\perp &\Leftrightarrow \forall \xi \in \mathcal{H}_\pi, \ \langle b, \partial\xi \rangle_\mu = 0 \\
&\Leftrightarrow \forall \xi \in \mathcal{H}_\pi, \ \int_G \langle b(h), \pi(h)\xi - \xi \rangle \, \mathrm{d}\mu(h) = 0 \\
&\Leftrightarrow \forall \xi \in \mathcal{H}_\pi, \ \int_G \langle \pi(h)^* b(h) - b(h), \xi \rangle \, \mathrm{d}\mu(h) = 0 \\
&\Leftrightarrow \forall \xi \in \mathcal{H}_\pi, \ \int_G \langle -b(h^{-1}) - b(h), \xi \rangle \, \mathrm{d}\mu(h) = 0
\end{aligned}$$

$$\begin{aligned}&\Leftrightarrow \forall \xi \in \mathcal{H}_\pi,\ \langle -2\, b(\mu), \xi\rangle = 0\\ &\Leftrightarrow b(\mu) = 0\\ &\Leftrightarrow b \in \mathrm{Har}_\mu(G,\pi).\end{aligned}$$

Then $(\mathrm{B}^1(G,\pi))^\perp = \mathrm{Har}_\mu(G,\pi)$. □

Remark 4.9. The proof of Theorem 4.7 actually shows the following more general result. Let $\mu \in \mathrm{Prob}(G)$ be any symmetric Borel probability measure on G that satisfies conditions (i) and (ii) in Terminology 4.6. Assume that for every $b \in \mathrm{Z}^1(G,\pi)$, we have $\|b\|_\mu < +\infty$. Then $(\mathrm{Z}^1(G,\pi), \langle \cdot\,,\cdot\rangle_\mu)$ is a Hilbert space.

3. Shalom's characterization of property (T)

The main result of this section is the following characterization of property (T) for *compactly generated* lcsc groups due to Shalom [**Sh99**] (see also [**Mo95, KS96**] for the case of finitely generated discrete groups).

Theorem 4.10 (Shalom [**Sh99**]). *Let G be any compactly generated lcsc group. If G does not have property* (T)*, then there exists a strongly continuous unitary representation $\pi : G \to \mathcal{U}(\mathcal{H}_\pi)$ such that $\overline{\mathrm{H}}^1(G,\pi) \neq 0$.*

Before proving Theorem 4.10, we discuss the *ultraproduct* construction in the setting of strongly continuous unitary representations. Let $\mathcal{U} \in \beta(\mathbb{N}) \setminus \mathbb{N}$ be any nonprincipal ultrafilter and $\pi : G \to \mathcal{U}(\mathcal{H}_\pi)$ any strongly continuous unitary representation. Set

$$\begin{aligned}\mathfrak{I}_\mathcal{U} &\doteq \left\{(\xi_n)_n \in \ell^\infty(\mathbb{N},\mathcal{H}_\pi) \mid \lim_{n\to\mathcal{U}} \|\xi_n\| = 0\right\}\\ \mathfrak{H}_\mathcal{U} &\doteq \ell^\infty(\mathbb{N},\mathcal{H}_\pi)/\mathfrak{I}_\mathcal{U}.\end{aligned}$$

For every $(\xi_n)_n \in \ell^\infty(\mathbb{N},\mathcal{H}_\pi)$, we simply denote by $(\xi_n)_\mathcal{U}$ its image in $\mathfrak{H}_\mathcal{U}$. Endowed with the sesquilinear form $\langle (\xi_n)_\mathcal{U}, (\eta_n)_\mathcal{U}\rangle_{\mathfrak{H}_\mathcal{U}} = \lim_{n\to\mathcal{U}} \langle \xi_n, \eta_n\rangle$, the vector space $\mathfrak{H}_\mathcal{U}$ is a Hilbert space. We call $\mathfrak{H}_\mathcal{U}$ the *ultraproduct* Hilbert space.

When G is discrete, it is straightforward to define the *ultraproduct* unitary representation $\pi_\mathcal{U} : G \to \mathcal{U}(\mathfrak{H}_\mathcal{U})$ by the formula

$$\forall g \in G, \forall (\xi_n)_\mathcal{U} \in \mathfrak{H}_\mathcal{U}, \quad \pi_\mathcal{U}(g)(\xi_n)_\mathcal{U} = (\pi(g)\xi_n)_\mathcal{U}.$$

When G is not discrete, the above formula still makes sense but $\pi_\mathcal{U}$ need not be continuous on $\mathfrak{H}_\mathcal{U}$. We will now define a $\pi_\mathcal{U}$-invariant closed subspace $\mathfrak{H}_{\mathcal{U},\pi} \subset \mathfrak{H}_\mathcal{U}$ on which the unitary representation $\pi_\mathcal{U}$ is strongly continuous.

We say that a sequence $(\xi_n)_n \in \ell^\infty(\mathbb{N},\mathcal{H}_\pi)$ is $(\mathcal{U},\pi)$-*equicontinuous* if for every $\varepsilon > 0$, there exists a neighborhood $O \subset G$ of $e \in G$ such that $\{n \in \mathbb{N} \mid \forall h \in O, \|\pi(h)\xi_n - \xi_n\| < \varepsilon\} \in \mathcal{U}$. Set

$$\begin{aligned}\mathfrak{E}_{\mathcal{U},\pi} &\doteq \{(\xi_n)_n \in \ell^\infty(\mathbb{N},\mathcal{H}_\pi) \mid (\xi_n)_n \text{ is } (\mathcal{U},\pi)\text{-equicontinuous}\}\\ \mathfrak{H}_{\mathcal{U},\pi} &\doteq \mathfrak{E}_{\mathcal{U},\pi}/\mathfrak{I}_\mathcal{U}.\end{aligned}$$

Observe that $\mathfrak{H}_{\mathcal{U},\pi} \subset \mathfrak{H}_{\mathcal{U}}$ is a closed subspace.

LEMMA 4.11. *For every $g \in G$ and every $(\xi_n)_n \in \mathfrak{E}_{\mathcal{U},\pi}$, $(\pi(g)\xi_n)_n \in \mathfrak{E}_{\mathcal{U},\pi}$. Then the closed subspace $\mathfrak{H}_{\mathcal{U},\pi} \subset \mathfrak{H}_{\mathcal{U}}$ is $\pi_{\mathcal{U}}$-invariant and the unitary representation $\pi_{\mathcal{U}} : G \to \mathcal{U}(\mathfrak{H}_{\mathcal{U},\pi})$ is strongly continuous.*

PROOF. Let $g \in G$ and $(\xi_n)_n \in \mathfrak{E}_{\mathcal{U},\pi}$. For every $\varepsilon > 0$, there exists a neighborhood $O \subset G$ of $e \in G$ such that $\{n \in \mathbb{N} \mid \forall h \in O, \|\pi(h)\xi_n - \xi_n\| < \varepsilon\} \in \mathcal{U}$. Set $O_g = gOg^{-1}$ and observe that $O_g \subset G$ is still a neighborhood of $e \in G$. We have

$$\begin{aligned}\{n \in \mathbb{N} \mid \forall h \in O_g, \|\pi(h)\,\pi(g)\xi_n - \pi(g)\xi_n\| < \varepsilon\} \\ = \{n \in \mathbb{N} \mid \forall h \in O_g, \|\pi(g)^*\pi(h)\pi(g)\xi_n - \xi_n\| < \varepsilon\} \\ = \{n \in \mathbb{N} \mid \forall h \in O, \|\pi(h)\xi_n - \xi_n\| < \varepsilon\} \in \mathcal{U}.\end{aligned}$$

This shows that $(\pi(g)\xi_n)_n \in \mathfrak{E}_{\mathcal{U},\pi}$. This further implies that $\pi_{\mathcal{U}} : G \to \mathcal{U}(\mathfrak{H}_{\mathcal{U},\pi})$ is a well-defined unitary representation.

Secondly, we prove that $\pi_{\mathcal{U}} : G \to \mathcal{U}(\mathfrak{H}_{\mathcal{U},\pi})$ is strongly continuous. Let $(\xi_n)_n \in \mathfrak{E}_{\mathcal{U},\pi}$ and set $\xi = (\xi_n)_{\mathcal{U}} \in \mathfrak{H}_{\mathcal{U},\pi}$. For every $\varepsilon > 0$, there exists a neighborhood $O \subset G$ of $e \in G$ such that $\{n \in \mathbb{N} \mid \forall h \in O, \|\pi(h)\xi_n - \xi_n\| < \varepsilon\} \in \mathcal{U}$. This implies that for every $h \in O$, we have

$$\|\pi_{\mathcal{U}}(h)\xi - \xi\| = \lim_{n \to \mathcal{U}} \|\pi(h)\xi_n - \xi_n\| \leq \varepsilon.$$

This shows that $\pi_{\mathcal{U}} : G \to \mathcal{U}(\mathfrak{H}_{\mathcal{U},\pi})$ is strongly continuous at $e \in G$ and so $\pi_{\mathcal{U}} : G \to \mathcal{U}(\mathfrak{H}_{\mathcal{U},\pi})$ is strongly continuous. □

We are now ready to prove Theorem 4.10.

PROOF OF THEOREM 4.10. We follow the lines of the proof given by Erschler–Ozawa [**EO16**, Section 4]. Fix a cohomologically adapted symmetric Borel probability measure $\mu \in \mathrm{Prob}(G)$ as in Terminology 4.6. We moreover assume that μ is compactly supported, $\mu = \mu_0 * \mu_0$ for some symmetric Borel probability measure $\mu_0 \in \mathrm{Prob}(G)$ and $e \in \mathrm{supp}(\mu)$. Using Theorem 4.7, it suffices to show that there exists a nonzero μ-harmonic cocycle for some strongly continuous unitary representation.

Since G does not have property (T), there exists a strongly continuous unitary representation $\pi : G \to \mathcal{U}(\mathcal{H}_\pi)$ that is ergodic and that has almost invariant vectors. Consider the bounded operator $\pi(\mu) \in \mathbb{B}(\mathcal{H}_\pi)$ defined by the formula

$$\forall \xi, \eta \in \mathcal{H}_\pi, \quad \langle \pi(\mu)\xi, \eta \rangle = \int_G \langle \pi(g)\xi, \eta \rangle \, \mathrm{d}\mu(g). \tag{4.7}$$

We will simply write $\pi(\mu) = \int_G \pi(g) \, \mathrm{d}\mu(g)$. Since μ_0 is symmetric and since $\mu = \mu_0 * \mu_0$, we have $\pi(\mu) = \pi(\mu_0)\pi(\mu_0) = \pi(\mu_0)^*\pi(\mu_0)$. It follows that $\pi(\mu)$ is a positive selfadjoint bounded operator such that $\|\pi(\mu)\| \leq 1$. Then its spectrum satisfies $\sigma(\pi(\mu)) \subset [0, 1]$. Since π is ergodic, 1 is not an eigenvalue for $\pi(\mu)$. Indeed, if $\eta \in \mathcal{H}_\pi$ satisfies $\pi(\mu)\eta = \eta$, then (4.7) implies that

$$\int_G \|\pi(g)\eta - \eta\|^2 \, \mathrm{d}\mu(g) = \|\pi(\mu)\eta - \eta\|^2 = 0.$$

This implies that $\pi(g)\eta = \eta$ for μ-almost every $g \in G$. Since π is strongly continuous, we obtain $\pi(g)\eta = \eta$ for every $g \in \operatorname{supp}(\mu)$. Since $G = \bigcup_{n \geq 1} \operatorname{supp}(\mu)^n$, it follows that $\pi(g)\eta = \eta$ for every $g \in G$ and so $\eta = 0$. Since π has almost invariant vectors, (4.7) and Lebesgue's dominated convergence theorem imply that $1 \in \sigma(\pi(\mu))$. More precisely, we have the following useful result.

Claim 4.12. Set $T = \pi(\mu)$. Then there exists a unit vector $\xi \in \mathcal{H}_\pi$ and a Borel probability measure $\nu \in \operatorname{Prob}([0,1])$ such that $1 \in \operatorname{supp}(\nu)$, $\nu(\{1\}) = 0$, and

$$\forall f \in \mathrm{B}([0,1]), \quad \int_0^1 f(t) \, \mathrm{d}\nu(t) = \langle f(T)\xi, \xi \rangle. \tag{4.8}$$

Indeed, using the assumptions, for every $n \geq 1$, the spectral projection $p_n = \mathbf{1}_{[1-1/n,1]}(T)$ is nonzero. Then for every $n \geq 1$, $p_n^\perp(\mathcal{H}_\pi)$ is a proper closed subspace with empty interior. Baire's theorem implies that $\bigcup_{n \geq 1} p_n^\perp(\mathcal{H}_\pi)$ is a proper subspace of $\mathcal{H}_\pi$. Choose a unit vector $\xi \in \mathcal{H}_\pi \setminus \bigcup_{n \geq 1} p_n^\perp(\mathcal{H}_\pi)$. Then we have $p_n\xi \neq 0$ for every $n \geq 1$. Denote by $\nu \in \operatorname{Prob}([0,1])$ the unique Borel probability measure satisfying (4.8). Then $\nu([1-1/n,1]) = \langle p_n\xi, \xi \rangle = \|p_n\xi\|^2 \neq 0$ and so $1 \in \operatorname{supp}(\nu)$. Since $\mathbf{1}_{\{1\}}(T) = 0$, we have $\nu(\{1\}) = 0$. This finishes the proof of Claim 4.12.

Next, for every $n \in \mathbb{N}$, consider the coboundary $c_n : G \to \mathcal{H}_\pi : g \mapsto T^{n/2}\xi - \pi(g)T^{n/2}\xi$. We have

$$\begin{aligned}
\|c_n\|_\mu^2 &= \int_G \|c_n(g)\|^2 \, \mathrm{d}\mu(g) \\
&= 2\left(\|T^{n/2}\xi\|^2 - \langle T^{n/2}\xi, T^{(n+2)/2}\xi \rangle\right) \\
&= 2\langle T^n(1-T)\xi, \xi \rangle \\
&= 2\int_0^1 t^n(1-t) \, \mathrm{d}\nu(t) \doteq 2\,\gamma(n).
\end{aligned}$$

We will need the following elementary technical result.

Claim 4.13. The following assertions hold:

(i) The sequence $(\gamma(n))_n$ is decreasing and $\lim_n \gamma(n) = 0$.
(ii) The sequence $(\frac{\gamma(n+1)}{\gamma(n)})_n$ is increasing and $\lim_n \frac{\gamma(n+1)}{\gamma(n)} = 1$.

Indeed, item (i) follows from Lebesgue's dominated convergence theorem. For item (ii), first note that for every $n \in \mathbb{N}$, $\gamma(n) > 0$. Indeed, otherwise ν would be supported on $\{0,1\}$, which is absurd by construction.

Next, Cauchy–Schwarz's inequality implies

$$\gamma(n+1) = \int_0^1 t^{n/2}(1-t)^{1/2} \cdot t^{(n+2)/2}(1-t)^{1/2} \, \mathrm{d}\nu(t) \leq \gamma(n)^{1/2} \cdot \gamma(n+2)^{1/2}$$

and so the sequence $(\frac{\gamma(n+1)}{\gamma(n)})_n$ is increasing. Denote by $\ell = \lim_n \frac{\gamma(n+1)}{\gamma(n)}$. By contradiction, assume that $\ell < 1$. Then every $n \in \mathbb{N}$, we have $\gamma(n) \leq \ell^n$ and the monotone convergence theorem implies that

$$\int_0^1 t^n \, \mathrm{d}\nu(t) = \sum_{k=n}^{\infty} \gamma(k) \leq \frac{1}{1-\ell} \, \ell^n.$$

This implies that $\nu(]\ell, 1]) = 0$, which contradicts the assumption that $1 \in \operatorname{supp}(\nu)$. This finishes the proof of Claim 4.13.

Next, for every $n \in \mathbb{N}$, consider the *normalized* coboundary $b_n : G \to \mathcal{H}_\pi : g \mapsto \frac{1}{\|c_n\|_\mu} c_n(g)$. We would like to define the map $b_\mathcal{U} : G \to \mathfrak{H}_{\mathcal{U},\pi} : g \mapsto (b_n(g))_\mathcal{U}$ and show that it is a cocycle for the ultraproduct representation $\pi_\mathcal{U}$. When G is discrete, this is straightforward and the reader can skip Claims 4.14 and 4.15. When G is arbitrary, we need to show that $b_\mathcal{U}$ is well-defined, namely that $(b_n(g))_n \in \mathfrak{E}_{\mathcal{U},\pi}$ for every $g \in G$, and that $b_\mathcal{U} : G \to \mathfrak{H}_{\mathcal{U},\pi}$ is continuous. As an intermediate step, we prove the following equicontinuity result for the family $(b_n)_n$.

Claim 4.14. For every compact subset $C \subset G$ such that $Q \subset C$, there exists a continuous function $\delta_C : G \to \mathbb{R}_+$, that is bounded on C, such that $\delta(e) = 0$ and for which

$$\forall n \in \mathbb{N}, \forall g \in C, \quad \|b_n(g)\| \leq \delta_C(g).$$

Indeed, set $f \doteq \frac{\mathrm{d}\mu}{\mathrm{d}m_G} \in \mathrm{L}^1(G, m_G)$ with $f \geq 0$ and $\|f\|_1 = 1$. Observe that for every $n \geq 2$ and every $g \in G$, we have

$$\begin{aligned}
-\int_G & \left(f(h) - f(g^{-1}h)\right) c_{n-2}(h) \, \mathrm{d}m_G(h) \\
&= -\int_G c_{n-2}(h) \, \mathrm{d}\mu(h) + \int_G c_{n-2}(gh) \, \mathrm{d}\mu(h) \\
&= (1 - \pi(g)) \int_G \pi(h) \, T^{(n-2)/2} \xi \, \mathrm{d}\mu(h) \\
&= c_n(g).
\end{aligned}$$

Set $K = C \cdot \operatorname{supp}(\mu)$ and observe that $K \subset G$ is a compact subset such that $Q \subset K$. By Theorem 4.7, there exists $\kappa > 0$ such that $p_K(c) \leq \kappa \, \|c\|_\mu$ for every $c \in \mathrm{Z}^1(G, \pi)$. Then for $n \geq 2$ and every $g \in Q$, we have

$$\begin{aligned}
\|b_n(g)\| = \frac{\|c_n(g)\|}{\|c_n\|_\mu} &\leq \frac{p_K(c_{n-2})}{\|c_n\|_\mu} \cdot \|f - \lambda(g) f\|_1 \\
&\leq \kappa \, \frac{\|c_{n-2}\|_\mu}{\|c_n\|_\mu} \cdot \|f - \lambda(g) f\|_1
\end{aligned}$$

$$= \kappa \left(\frac{\gamma(n-2)}{\gamma(n)} \right)^{1/2} \cdot \|f - \lambda(g) f\|_1.$$

Claim 4.13 implies that the sequence $(\left(\frac{\gamma(n-2)}{\gamma(n)}\right)^{1/2})_n$ is bounded. Moreover, the left translation action $\lambda : G \curvearrowright \mathrm{L}^1(G)$ is continuous. This finishes the proof of Claim 4.14.

Claim 4.15. The following assertions hold:

(i) For every $g \in G$, $(b_n(g))_n \in \mathfrak{E}_{\mathcal{U},\pi}$.

(ii) The well-defined map $b_{\mathcal{U}} : G \to \mathfrak{H}_{\mathcal{U},\pi} : g \mapsto (b_n(g))_{\mathcal{U}}$ is continuous and is a cocycle for the ultraproduct representation $\pi_{\mathcal{U}}$.

Indeed, for item (i), let $g \in G$ be any element and $\varepsilon > 0$. Claim 4.14 implies that $(b_n(g))_n \in \ell^\infty(\mathbb{N}, \mathcal{H}_\pi)$ and that there exists a compact neighboorhood $O \subset G$ of $e \in G$ such that

$$\sup \left\{ \|b_n(h)\| + \|b_n(g^{-1}hg)\| \mid n \in \mathbb{N}, h \in O \right\} < \varepsilon.$$

For every $n \in \mathbb{N}$ and every $h \in O$, we have

$$\begin{aligned} \|\pi(h) b_n(g) - b_n(g)\| &= \|b_n(hg) - b_n(h) - b_n(g)\| \\ &= \|b_n(g\, g^{-1}hg) - b_n(h) - b_n(g)\| \\ &= \|b_n(g\, g^{-1}hg) - b_n(h) - b_n(g)\| \\ &= \|\pi(g) b_n(g^{-1}hg) - b_n(h)\| \\ &\leq \|b_n(g^{-1}hg)\| + \|b_n(h)\| < \varepsilon. \end{aligned}$$

This implies that $(b_n(g))_n \in \mathfrak{E}_{\mathcal{U},\pi}$.

For item (ii), it is clear that $b_{\mathcal{U}}$ satisfies the 1-cocycle relation (4.1) for $\pi_{\mathcal{U}}$. Moreover, Claim 4.14 implies that $b_{\mathcal{U}}$ is continuous at $e \in G$. Thus, $b_{\mathcal{U}}$ is continuous and is a cocycle for the ultraproduct representation $\pi_{\mathcal{U}}$. This finishes the proof of Claim 4.15.

Claim 4.16. The cocycle $b_{\mathcal{U}} : G \to \mathfrak{H}_{\mathcal{U},\pi}$ is nonzero and μ-harmonic.

Indeed, applying Claim 4.14 to $C \doteq Q \cdot \mathrm{supp}(\mu)$, we have

$$\begin{aligned} \|b_{\mathcal{U}}\|_\mu^2 &= \int_G \|b_{\mathcal{U}}(h)\|^2 \, \mathrm{d}\mu(h) \\ &= \int_G \lim_{n \to \mathcal{U}} \|b_n(h)\|^2 \, \mathrm{d}\mu(h) \\ &= \lim_{n \to \mathcal{U}} \int_G \|b_n(h)\|^2 \, \mathrm{d}\mu(h) \quad \text{(by Claim 4.14)} \\ &= 1. \end{aligned}$$

This shows that $b_{\mathcal{U}}$ is nonzero. Moreover, using Claim 4.13, we have

$$\| \int_G b_n(h) \, \mathrm{d}\mu(h) \|^2 = \frac{1}{2\gamma(n)} \| \int_G c_n(h) \, \mathrm{d}\mu(h) \|^2$$

$$
\begin{aligned}
&= \frac{1}{2\gamma(n)} \|T^{n/2}\xi - T^{(n+2)/2}\xi\|^2 \\
&= \frac{1}{2\gamma(n)} \int_0^1 (t^n - 2t^{n+1} + t^{n+2})\, \mathrm{d}\nu(t) \\
&= \frac{\gamma(n) - \gamma(n+1)}{2\gamma(n)} \to 0.
\end{aligned}
$$

Then for every $(\xi_n)_n \in \mathfrak{E}_{\mathcal{U},\pi}$, applying Claim 4.14 to $C \doteq Q \cdot \mathrm{supp}(\mu)$, we obtain

$$
\begin{aligned}
\langle \int_G b_{\mathcal{U}}(h)\, \mathrm{d}\mu(h), (\xi_n)_{\mathcal{U}} \rangle &= \int_G \langle b_{\mathcal{U}}(h), (\xi_n)_{\mathcal{U}} \rangle \, \mathrm{d}\mu(h) \\
&= \int_G \lim_{n \to \mathcal{U}} \langle b_n(h), \xi_n \rangle \, \mathrm{d}\mu(h) \\
&= \lim_{n \to \mathcal{U}} \int_G \langle b_n(h), \xi_n \rangle \, \mathrm{d}\mu(h) \quad \text{(by Claim 4.14)} \\
&= \lim_{n \to \mathcal{U}} \langle \int_G b_n(h)\, \mathrm{d}\mu(h), \xi_n \rangle = 0.
\end{aligned}
$$

This shows that $b_{\mathcal{U}}$ is μ-harmonic and finishes the proof of Claim 4.16.

Combining Theorem 4.7 and Claim 4.16, we obtain $\overline{\mathrm{H}}^1(G, \pi_{\mathcal{U}}) \neq 0$. This finishes the proof of Theorem 4.10. □

4. Induction and reduced cohomology

Let G be any compactly generated lcsc group and $\Gamma < G$ any lattice. Set $X \doteq G/\Gamma$ and denote by $\nu \in \mathrm{Prob}(X)$ the unique G-invariant Borel probability measure on X. For every Borel fundamental domain $\mathcal{F} \subset G$, we may choose a Borel section $\sigma : X \to \mathcal{F}$ as in Corollary 1.12. For every $g \in G$ and every $x \in G/\Gamma$, denote by $\tau(g, x) \in \Gamma$ the unique element in Γ such that $g\sigma(x) = \sigma(gx)\tau(g, x)$. The map $\tau : G \times X \to \Gamma$ is Borel and satisfies the 1-cocycle relation (2.1). Denote by m_G the unique Haar measure on G such that $\sigma_*\nu = m_G|_{\mathcal{F}}$.

From now on, we assume that the lattice $\Gamma < G$ is finitely generated. This is always the case when $\Gamma < G$ is uniform (see Proposition 1.14) or when G has property (T) (see Propositions 2.26 and 2.28). Fix a finite symmetric generating set $S \subset \Gamma$ and define the word length $\ell_S : \Gamma \to \mathbb{N}$ on Γ associated with S by the formula

$$\forall \gamma \in \Gamma, \quad \ell_S(\gamma) \doteq \min\{n \in \mathbb{N} \mid \gamma \in S^n\}.$$

Definition 4.17 (Shalom [Sh99]). We say that the lattice $\Gamma < G$ is L^2-*integrable* if there exists a Borel fundamental domain $\mathcal{F} \subset G$ for which the associated 1-cocycle $\tau : G \times X \to \Gamma$ satisfies the L^2-integrability condition:

$$\forall g \in G, \quad \int_X \ell_S(\tau(g, g^{-1}x))^2 \, \mathrm{d}\nu(x) < +\infty. \tag{4.9}$$

Any uniform lattice $\Gamma < G$ is L^2-integrable. Indeed, in that case, using Proposition 1.11(ii) we may choose a relatively compact Borel fundamental domain $\mathcal{F} \subset G$. Then for every $g \in G$, the subset $\mathcal{F}^{-1} g \mathcal{F}$ is relatively compact in G and so $\tau(g, X) \subset \mathcal{F}^{-1} g \mathcal{F} \cap \Gamma$ is finite. Then (4.9) is satisfied.

The next theorem due to Shalom provides examples of nonuniform L^2-integrable lattices in locally compact groups.

THEOREM 4.18 (Shalom [**Sh99**]). *The following examples of nonuniform lattices are* L^2*-integrable:*

(i) *For every* $d \geq 3$, $\mathrm{SL}_d(\mathbb{Z}) < \mathrm{SL}_d(\mathbb{R})$.
(ii) *For every* $d \geq 2$ *and every square-free integer* $q \geq 2$,

$$\mathrm{SL}_d(\mathbb{Z}[\sqrt{q}]) < \mathrm{SL}_d(\mathbb{R}) \times \mathrm{SL}_d(\mathbb{R}).$$

(iii) *For every* $d \geq 2$ *and every prime* $p \in \mathcal{P}$,

$$\mathrm{SL}_d(\mathbb{Z}[p^{-1}]) < \mathrm{SL}_d(\mathbb{R}) \times \mathrm{SL}_d(\mathbb{Q}_p).$$

More generally, it is proven in [**Sh99**, §2] that irreducible lattices in *higher rank* semisimple algebraic groups are L^2-integrable.

PROOF. We only explain the proof of item (i). We refer to [**Sh99**, §2] for the proof in the general case of irreducible lattices in higher rank semisimple algebraic groups that covers items (ii) and (iii).

Let $d \geq 3$ and set $\Gamma \doteq \mathrm{SL}_d(\mathbb{Z}) < \mathrm{SL}_d(\mathbb{R}) \doteq G$ and $X \doteq G/\Gamma$. Following Theorem 1.19, choose $t \geq \frac{2}{\sqrt{3}}$ and $u \geq \frac{1}{2}$ so that $G = \mathfrak{S}_{t,u} \cdot \Gamma$ where $\mathfrak{S}_{t,u} \subset G$ is a Siegel domain of finite Haar measure. Then we may choose a Borel fundamental domain $\mathcal{F} \subset \mathfrak{S}_{t,u} \subset G$ (see [**Zi84**, Corollary A.6]).

Denote by $S \subset \Gamma$ the finite symmetric set of all elementary matrices defined as follows

$$S \doteq \{E_{ij}(\pm 1) \mid 1 \leq i \neq j \leq d\}.$$

Note that S is a generating set for Γ. Consider the length function $\ell_S : \Gamma \to \mathbb{N}$ on Γ associated with S. On $\mathbb{R}^d$, consider the canonical L^2-norm $\|\cdot\|_2$ and define

$$\forall g \in G, \quad \|g\| \doteq \sup\left\{\|gv\|_2 \mid v \in \mathbb{R}^d, \|v\|_2 \leq 1\right\}.$$

Using the Cartan decomposition $G = K \cdot A^+ \cdot K$ from Lemma 2.37, we have $\|g\| \geq 1$ and $\|g^{-1}\| \leq \|g\|^{d-1}$ for every $g \in G$.

The main result of [**LMR96**] implies that the lengths ℓ_S and $\log(\|\cdot\|)$ are coarse Lipschitz equivalent on Γ. In particular, there are constants $a, b > 0$ such that $\ell_S(\gamma) \leq a \log(\|\gamma\|) + b$ for every $\gamma \in \Gamma$. Then for every $g \in G$ and every $x \in X$, we have

$$\begin{aligned}
\ell_S(\tau(g, g^{-1}x)) &\leq a \log(\|\tau(g, g^{-1}x)\|) + b \\
&\leq a \log(\|\sigma(x)^{-1}\|) + a \log(\|g\|) + \log(\|\sigma(g^{-1}x)\|) + b \\
&\leq a(d-1) \log(\|\sigma(x)\|) + a \log(\|g\|) + \log(\|\sigma(g^{-1}x)\|) + b.
\end{aligned}$$

Next observe that for every $g \in G$, we have

$$\int_X \log(\|\sigma(g^{-1}x)\|)^2 \,\mathrm{d}\nu(x) = \int_X \log(\|\sigma(x)\|)^2 \,\mathrm{d}\nu(x) = \int_{\mathcal{F}} \log(\|y\|)^2 \,\mathrm{d}m_G(y).$$

In order to prove that $\Gamma < G$ is L^2-integrable, it suffices to prove that

$$\int_{\mathcal{F}} \log(\|y\|)^2 \,\mathrm{d}m_G(y) < +\infty.$$

For this, recall that $\mathcal{F} \subset \mathfrak{S}_{t,u}$ and that $\mathfrak{S}_{t,u} = K \cdot A_t \cdot N_u$ where $t \geq \frac{2}{\sqrt{3}}$ and $u \geq \frac{1}{2}$. Since K and N_u are both compact in $\mathrm{SL}_d(\mathbb{R})$ and since for every $a = \mathrm{diag}(\lambda_1, \ldots, \lambda_d) \in A_t$, we have $1 \leq \|a\| \leq t^{d-1}\lambda_d$, using Lemma 1.21, it suffices to prove that

$$\int_{A_t} (\log \lambda_d)^2 \prod_{1 \leq i < j \leq d} \frac{\lambda_i}{\lambda_j} \,\mathrm{d}a < +\infty.$$

Observe that the map

$$\Theta : A \to \mathbb{R}^{d-1} : \mathrm{diag}(\lambda_1, \ldots, \lambda_d) \mapsto \left(\log \frac{\lambda_2}{\lambda_1}, \ldots, \log \frac{\lambda_d}{\lambda_{d-1}}\right)$$

is a topological group isomorphism. We may choose the Haar measure $\mathrm{d}a$ on A that is the pushforward of the Lebesgue measure on $\mathbb{R}^{d-1}$ by Θ^{-1}. For every $a = \mathrm{diag}(\lambda_1, \ldots, \lambda_d) \in A_t$, letting

$$\left(\log \frac{\lambda_2}{\lambda_1}, \ldots, \log \frac{\lambda_d}{\lambda_{d-1}}\right) = (s_1, \ldots, s_{d-1})$$

we have

$$\begin{aligned}
(\log \lambda_d)^2 &= \frac{1}{d^2}\left(\log \frac{\lambda_d}{\lambda_1} + \cdots + \log \frac{\lambda_d}{\lambda_{d-1}}\right)^2 \\
&= \frac{1}{d^2}\left(\sum_{k=1}^{d-1} k s_k\right)^2 \\
&\leq \frac{d-1}{d^2} \sum_{k=1}^{d-1} k^2 s_k^2.
\end{aligned}$$

A simple calculation as in Claim 1.22 shows that for every $1 \leq k \leq d-1$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^{d-1}} s_k^2 \prod_{1 \leq i < j \leq d} \exp(-(s_i + \cdots + s_{j-1})) \mathbf{1}_{\{s_1, \ldots, s_{d-1} \geq -\log t\}} \,\mathrm{d}s_1 \cdots \mathrm{d}s_{d-1} \\
&= \int_{-\log t}^{+\infty} s_k^2 \exp(-k(d-k)s_k) \,\mathrm{d}s_k \cdot \prod_{j \neq k} \int_{-\log t}^{+\infty} \exp(-j(d-j)s_j) \,\mathrm{d}s_j < +\infty.
\end{aligned}$$

Altogether, this implies that $\int_{\mathcal{F}} \log(\|y\|)^2 \,\mathrm{d}m_G(y) < +\infty$ and hence $\Gamma < G$ is L^2-integrable. $\square$

REMARK 4.19. We point out that the assumption that $d \geq 3$ in Theorem 4.18(i) is necessary. Indeed, when $d = 2$, the lengths ℓ_S and $\log(\|\cdot\|)$ are not Lipschitz equivalent on $\mathrm{SL}_2(\mathbb{Z}) < \mathrm{SL}_2(\mathbb{R})$. Indeed, set

$$\gamma \doteq \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) < \mathrm{SL}_2(\mathbb{R}).$$

Then for every $n \geq 1$, $\ell_S(\gamma^n) = n$ while $\log(\|\gamma^n\|) = O(\log(n))$ since

$$\gamma^n = \begin{pmatrix} n & 0 \\ 0 & n^{-1} \end{pmatrix} \begin{pmatrix} 1 & n^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} n^{-1} & 0 \\ 0 & n \end{pmatrix}.$$

REMARK 4.20. The proof of Theorem 4.18(i) actually shows the following *stronger* L^2-integrability condition:

$$\forall C \subset G \text{ compact subset}, \quad \sup_{g \in C} \int_X \ell_S(\tau(g, g^{-1}x))^2 \, \mathrm{d}\nu(x) < +\infty. \tag{4.10}$$

From now on, we fix a finitely generated L^2-integrable lattice $\Gamma < G$. For every unitary representation $\pi : \Gamma \to \mathcal{U}(\mathcal{H}_\pi)$ with $\mathcal{H}_\pi$ separable, we denote by $\widehat{\pi} : G \to \mathcal{U}(\mathcal{H}_{\widehat{\pi}})$ the induced representation (see Chapter 2). We define the *induction* from Γ to G for cocycles using the first viewpoint on induction.

DEFINITION 4.21 (Shalom [**Sh99**]). Let $b \in \mathrm{Z}^1(\Gamma, \pi)$ be any cocycle. Define the *induced cocycle* $\widehat{b} : G \to \mathcal{H}_{\widehat{\pi}}$ by the formula

$$\forall g \in G, \forall x \in X, \quad \widehat{b}(g)(x) = b(\tau(g, g^{-1}x)) \in \mathcal{H}_\pi.$$

We need to check that the map $\widehat{b} : G \to \mathcal{H}_{\widehat{\pi}}$ is indeed a cocycle for $\widehat{\pi}$, that is, $b \in \mathrm{Z}^1(G, \widehat{\pi})$. As before, set $p_S(b) \doteq \sup \{\|b(\gamma)\| \mid \gamma \in S\}$. Note that for every $g \in G$, we have $\widehat{b}(g) \in \mathcal{H}_{\widehat{\pi}}$ since

$$\begin{aligned} \|\widehat{b}(g)\|_\nu^2 &= \int_X \|b(\tau(g, g^{-1}x))\|^2 \, \mathrm{d}\nu(x) \\ &\leq \int_X \ell_S(\tau(g, g^{-1}x))^2 \, \mathrm{d}\nu(x) \cdot p_S(b)^2 < +\infty. \end{aligned}$$

Moreover, $\widehat{b}$ satisfies the cocycle relation (4.1) for $\widehat{\pi}$. Indeed, for all $g, h \in G$ and every $x \in X$, we have

$$\begin{aligned} \widehat{b}(gh)(x) &= b(\tau(gh, h^{-1}g^{-1}x)) \\ &= b(\tau(g, g^{-1}x)\, \tau(h, h^{-1}g^{-1}x)) \\ &= b(\tau(g, g^{-1}x)) + \pi(\tau(g, g^{-1}x)) b(\tau(h, h^{-1}g^{-1}x)) \\ &= \left(\widehat{b}(g) + \widehat{\pi}(g)\widehat{b}(h)\right)(x). \end{aligned}$$

By Fubini's theorem, the function $G \to \mathbb{R}_+ : g \mapsto \|\widehat{b}(g)\|_\nu$ is measurable. Since $\mathcal{H}_{\widehat{\pi}} = \mathrm{L}^2(X, \nu) \otimes \mathcal{H}_\pi$ is separable, Lemma 4.2 implies that $\widehat{b} \in \mathrm{Z}^1(G, \widehat{\pi})$.

As it will be useful later on, we also define the induced cocycle using the second viewpoint on induction. With respect to the second viewpoint, the induced cocycle $\widehat{b} : G \to \mathcal{H}_{\widehat{\pi}}$ is defined by the formula

$$\begin{aligned}\forall g, h \in G, \quad \widehat{b}(g)(h) &= \pi(\tau(h^{-1}, h\Gamma))\, b(\tau(g, g^{-1}h\Gamma)) \\ &= b(\tau(h^{-1}g, g^{-1}h\Gamma)) - b(\tau(h^{-1}, h\Gamma)).\end{aligned}$$

The main result of this section due to Shalom shows that the induction from Γ to G for cocycles yields an injective map in (reduced) cohomology.

Theorem 4.22 (Shalom [**Sh99, Sh03**]). *The induction map*

$$\mathcal{I} : \mathrm{Z}^1(\Gamma, \pi) \to \mathrm{Z}^1(G, \widehat{\pi}) : b \mapsto \widehat{b}$$

is continuous and satisfies

$$\mathcal{I}(\mathrm{B}^1(\Gamma, \pi)) \subset \mathrm{B}^1(G, \widehat{\pi}) \quad \textit{and} \quad \mathcal{I}(\overline{\mathrm{B}^1(\Gamma, \pi)}) \subset \overline{\mathrm{B}^1(G, \widehat{\pi})}.$$

Moreover, the canonical well-defined mappings

$$\mathrm{H}^1(\Gamma, \pi) \to \mathrm{H}^1(G, \widehat{\pi}) \quad \textit{and} \quad \overline{\mathrm{H}}^1(\Gamma, \pi) \to \overline{\mathrm{H}}^1(G, \widehat{\pi})$$

are both injective.

Proof. First, we prove that the induction map $\mathcal{I} : \mathrm{Z}^1(\Gamma, \pi) \to \mathrm{Z}^1(G, \widehat{\pi})$ is continuous. This is immediate if the lattice $\Gamma < G$ satisfies the stronger L^2-integrability condition as in (4.10). Indeed, in that case, for every $b \in \mathrm{Z}^1(\Gamma, \pi)$, we have

$$\begin{aligned} p_Q(\widehat{b})^2 &= \sup_{g \in Q} \|\widehat{b}(g)\|_\nu^2 \\ &= \sup_{g \in Q} \int_X \|\widehat{b}(g)(x)\|^2 \, \mathrm{d}\nu(x) \\ &= \sup_{g \in Q} \int_X \|b(\tau(g, g^{-1}x))\|^2 \, \mathrm{d}\nu(x) \\ &\leq \sup_{g \in Q} \int_X \ell_S(\tau(g, g^{-1}x))^2 \, \mathrm{d}\nu(x) \cdot p_S(b). \end{aligned}$$

This shows that the induction map $\mathcal{I} : (\mathrm{Z}^1(\Gamma, \pi), p_S) \to (\mathrm{Z}^1(G, \widehat{\pi}), p_Q)$ is continuous.

In general, choose a cohomologically adapted symmetric Borel probability measure $\mu \in \mathrm{Prob}(G)$ as in Terminology 4.6 such that we moreover have $\mu \sim m_G$. Fix a symmetric compact neighborhood $Q \subset G$ of $e \in G$ such that $G = \bigcup_{n \geq 1} Q^n$. By Theorem 4.7, we know that $(\mathrm{Z}^1(G, \widehat{\pi}), \langle \cdot , \cdot \rangle_\mu)$ is a Hilbert space and for every compact subset $K \subset G$ such that $Q \subset K$, the norms p_K and $\| \cdot \|_\mu$ are equivalent on $\mathrm{Z}^1(G, \widehat{\pi})$.

Denote by $\mu_0 \in \mathrm{Prob}(\Gamma)$ the pushforward measure of $\mu \otimes \nu \in \mathrm{Prob}(G \times X)$ under the Borel map $G \times X \to \Gamma : (g, x) \mapsto \tau(g, g^{-1}x)$.

Claim 4.23. The measure $\mu_0 \in \mathrm{Prob}(\Gamma)$ is symmetric and $\mathrm{supp}(\mu_0) = \Gamma$.

Indeed, using Fubini's theorem and the facts that μ is symmetric and that ν is G-invariant, for every $\gamma \in \Gamma$, we have

$$\begin{aligned}\mu_0(\gamma) &= (\mu \otimes \nu)(\{(g,x) \in G \times X \mid \tau(g, g^{-1}x) = \gamma\}) \\ &= (\mu \otimes \nu)(\{(g,x) \in G \times X \mid \tau(g^{-1}, x) = \gamma^{-1}\}) \\ &= (\mu \otimes \nu)(\{(g,x) \in G \times X \mid \tau(g, x) = \gamma^{-1}\}) \\ &= (\mu \otimes \nu)(\{(g,x) \in G \times X \mid \tau(g, g^{-1}x) = \gamma^{-1}\}) \\ &= \mu_0(\gamma^{-1}).\end{aligned}$$

Then μ_0 is symmetric.

Next, let $\gamma \in \Gamma$ be any element. Since $\mu \sim m_G$, the left translation action $G \curvearrowright (G, \mu)$ is nonsingular and transitive. Choose a countable dense subgroup $\Lambda < G$ and set $B = \Lambda \cdot \mathcal{F} \subset G$. Then $\mu(B) > 0$ and $\mu(hB \triangle B) = 0$ for every $h \in \Lambda$. Since $G \to \mathbb{R}_+ : h \mapsto \mu(hB \triangle B)$ is continuous, it follows that $\mu(hB \triangle B) = 0$ for every $h \in G$. Then we have $\mu(B) = 1$ and so $\mu(B \cap \mathcal{F}\gamma) = \mu(\mathcal{F}\gamma) > 0$. Thus, there exists $g \in \Lambda$ such that $\mu(g\mathcal{F} \cap \mathcal{F}\gamma) > 0$. By continuity, we may choose a neighborhood $U \subset G$ of $g \in G$ such that $\mu(h\mathcal{F} \cap \mathcal{F}\gamma) > 0$ for every $h \in U$. Then we have $m_G(\mathcal{F} \cap h^{-1}\mathcal{F}\gamma) > 0$ for every $h \in U$ and so

$$\begin{aligned}\mu_0(\gamma) = \mu_0(\gamma^{-1}) &= (\mu \otimes \nu)(\{(g,x) \in G \times X \mid \tau(g,x) = \gamma\}) \\ &= (\mu \otimes \nu)(\{(g,x) \in G \times X \mid g\sigma(x) \in \mathcal{F}\gamma\}) \\ &\geq (\mu \otimes \nu)(\{(g,x) \in U \times X \mid \sigma(x) \in g^{-1}\mathcal{F}\gamma\}) \\ &= \int_U \nu(\{x \in X \mid \sigma(x) \in g^{-1}\mathcal{F}\gamma\}) \, \mathrm{d}\mu(g) \\ &= \int_U m_G(\mathcal{F} \cap g^{-1}\mathcal{F}\gamma) \, \mathrm{d}\mu(g) > 0.\end{aligned}$$

This finishes the proof of Claim 4.23.

For every $b \in \mathrm{Z}^1(\Gamma, \pi)$, we obtain

$$\begin{aligned}\|b\|_{\mu_0}^2 &= \int_\Gamma \|b(\gamma)\|^2 \, \mathrm{d}\mu_0(\gamma) \\ &= \int_{G \times X} \|b(\tau(g, g^{-1}x))\|^2 \, \mathrm{d}\mu(g)\mathrm{d}\nu(x) \\ &= \int_G \left(\int_X \|\widehat{b}(g)(x)\|^2 \, \mathrm{d}\nu(x) \right) \mathrm{d}\mu(g) \\ &= \int_G \|\widehat{b}(g)\|_\nu^2 \, \mathrm{d}\mu(g) \\ &= \|\widehat{b}\|_\mu^2 < +\infty.\end{aligned}$$

From the above equality, we infer the following crucial fact. For every $b \in \mathrm{Z}^1(\Gamma, \pi)$, we have $\|b\|_{\mu_0} < +\infty$. The proof of Theorem 4.7 then shows that the space $(\mathrm{Z}^1(\Gamma, \pi), \langle \cdot , \cdot \rangle_{\mu_0})$ is a Hilbert space (see Remark 4.9). Since

$\operatorname{supp}(\mu_0) = \Gamma$ and since $S \subset \Gamma$ is finite, there exists $\kappa_1 > 0$ such that $p_S \leq \kappa_1 \|\cdot\|_{\mu_0}$. This means that the identity linear mapping

$$\iota : (\mathrm{Z}^1(\Gamma, \pi), \|\cdot\|_{\mu_0}) \to (\mathrm{Z}^1(\Gamma, \pi), p_S) : b \mapsto b$$

is continuous and bijective. Since both $(\mathrm{Z}^1(\Gamma, \pi), \|\cdot\|_{\mu_0})$ and $(\mathrm{Z}^1(\Gamma, \pi), p_S)$ are Banach spaces, the open mapping theorem (see [**Ru91**, Corollaries 2.12]) implies that exists a constant κ_2 such that $\|\cdot\|_{\mu_0} \leq \kappa_2 \, p_S$. In other words, the norms $\|\cdot\|_{\mu_0}$ and p_S are equivalent on $\mathrm{Z}^1(\Gamma, \pi)$. As we have seen, the induction map $\mathcal{I} : (\mathrm{Z}^1(\Gamma, \pi), \|\cdot\|_{\mu_0}) \to (\mathrm{Z}^1(G, \widehat{\pi}), \|\cdot\|_{\mu})$ is an isometry. The previous reasoning implies that the induction map $\mathcal{I} : (\mathrm{Z}^1(\Gamma, \pi), p_S) \to (\mathrm{Z}^1(G, \widehat{\pi}), p_K)$ is continuous for every compact subset $K \subset G$ such that $Q \subset K$.

We use the notation $\partial_\pi : \mathcal{H}_\pi \to \mathrm{B}^1(\Gamma, \pi)$ (resp. $\partial_{\widehat{\pi}} : \mathcal{H}_{\widehat{\pi}} \to \mathrm{B}^1(G, \widehat{\pi})$) for the coboundary map. For every $\xi \in \mathcal{H}_\pi$, we have $\mathcal{I}(\partial_\pi \xi) = \partial_{\widehat{\pi}}(\mathbf{1}_X \otimes \xi)$. This shows that $\mathcal{I}(\mathrm{B}^1(\Gamma, \pi)) \subset \mathrm{B}^1(G, \widehat{\pi})$. Since $\mathcal{I}$ is continuous, this further implies that $\mathcal{I}(\overline{\mathrm{B}^1(\Gamma, \pi)}) \subset \overline{\mathrm{B}^1(G, \widehat{\pi})}$. This shows that the canonical mappings

$$\mathrm{H}^1(\Gamma, \pi) \to \mathrm{H}^1(G, \widehat{\pi}) \quad \text{and} \quad \overline{\mathrm{H}}^1(\Gamma, \pi) \to \overline{\mathrm{H}}^1(G, \widehat{\pi})$$

are well-defined. It remains to prove that they are both injective.

In order to do that, we introduce the following *transfer operator* that was suggested to us by Narutaka Ozawa (see also [**Sh03**, p. 144]). We use the second viewpoint on induction for cocycles. Choose a relatively compact subset $C \subset \mathcal{F}$ such that $m_G(C) > 0$. Set $K \doteq Q \cup \bigcup_{\gamma \in S} \overline{C} \gamma \overline{C}^{-1} \subset G$ and note that $K \subset G$ is a compact subset such that $Q \subset K$. Define the mapping $\mathcal{T} : \mathrm{Z}^1(G, \widehat{\pi}) \to \mathrm{Z}^1(\Gamma, \pi)$ by the formula

$$\forall c \in \mathrm{Z}^1(G, \widehat{\pi}), \forall \gamma \in \Gamma, \quad \mathcal{T}(c)(\gamma) \doteq \frac{1}{m_G(C)^2} \int_{C^2} c(g\gamma h^{-1})(g) \, \mathrm{d}m_G^{\otimes 2}(g, h).$$

Claim 4.24. The following assertions hold:

(i) The transfer operator $\mathcal{T} : (\mathrm{Z}^1(G, \widehat{\pi}), p_K) \to (\mathrm{Z}^1(\Gamma, \pi), p_S)$ is well-defined and continuous.

(ii) For every $b \in \mathrm{Z}^1(\Gamma, \pi)$, we have

$$\mathcal{T}(\widehat{b}) = b.$$

(iii) For every $\eta \in \mathcal{H}_{\widehat{\pi}}$, we have

$$\mathcal{T}(\partial_{\widehat{\pi}} \eta) = \partial_\pi \xi \quad \text{where} \quad \xi = \frac{1}{m_G(C)} \int_C \eta(g) \, \mathrm{d}m_G(g) \in \mathcal{H}_\pi.$$

Proof of Claim 4.24. Keep the same notation as before.

(i) Let $c \in \mathrm{Z}^1(G, \widehat{\pi})$ and $\gamma \in \Gamma$. Firstly, note that the map $G \times G \to \mathcal{H}_\pi : (g, h) \mapsto c(g\gamma h^{-1})(g)$ is measurable. Next, we have

$$\frac{1}{m_G(C)^2} \int_C \int_C \|c(g\gamma h^{-1})(g)\|^2 \, \mathrm{d}m_G(g) \mathrm{d}m_G(h)$$

$$
\begin{aligned}
&= \frac{1}{m_G(C)^2}\int_C \left(\int_{C\gamma^{-1}g^{-1}} \|c(h^{-1})(g)\|^2 \,\mathrm{d}m_G(h)\right)\mathrm{d}m_G(g)\\
&\leq \frac{1}{m_G(C)^2}\int_{C\gamma^{-1}C^{-1}} \left(\int_C \|c(h^{-1})(g)\|^2 \,\mathrm{d}m_G(g)\right)\mathrm{d}m_G(h)\\
&\leq \frac{1}{m_G(C)^2}\int_{C\gamma^{-1}C^{-1}} \|c(h^{-1})\|^2_{\mathcal{H}_{\widehat{\pi}}} \,\mathrm{d}m_G(h)\\
&\leq \frac{m_G(C\gamma^{-1}C^{-1})}{m_G(C)^2}\, p_{\overline{C}\gamma\overline{C}^{-1}}(c)^2 < +\infty.
\end{aligned}
$$

This implies that $\mathcal{T}(c)(\gamma) \in \mathcal{H}_\pi$. Letting

$$
\kappa \doteq \frac{1}{m_G(C)^2}\sup\left\{m_G(C\gamma^{-1}C^{-1}) \mid \gamma \in S\right\} < +\infty,
$$

we have $\|\mathcal{T}(c)(\gamma)\| \leq \kappa\, p_K(c)$ for every $\gamma \in S$. This shows that $p_S(\mathcal{T}(c)) \leq \kappa\, p_K(c)$ for every $c \in \mathrm{Z}^1(G, \widehat{\pi})$.

We next prove that $\mathcal{T}(c) \in \mathrm{Z}^1(\Gamma, \pi)$. Indeed, using Fubini's theorem, for almost every $(g_1, g_2, g_3) \in G \times G \times G$, we have

$$
c(g_1g_2)(g_3) = c(g_1)(g_3) + (\widehat{\pi}(g_1)c(g_2))(g_3) = c(g_1)(g_3) + c(g_2)(g_1^{-1}g_3).
$$

Moreover, using Fubini's theorem, for every $\gamma \in \Gamma$ and almost every $(g_1, g_2) \in G \times G$, we have

$$
c(g_1)(g_2\gamma^{-1}) = \pi(\gamma)\, c(g_1)(g_2).
$$

These facts imply that for every $(\gamma_1, \gamma_2) \in \Gamma \times \Gamma$ and almost every $(g, h, k) \in G \times G \times G$, we have

$$
\begin{aligned}
c(g\gamma_1\gamma_2h^{-1})(g) &= c(g\gamma_1k^{-1}\, k\gamma_2h^{-1})(g)\\
&= c(g\gamma_1k^{-1})(g) + c(k\gamma_2h^{-1})(k\gamma_1^{-1})\\
&= c(g\gamma_1k^{-1})(g) + \pi(\gamma_1)\, c(k\gamma_2h^{-1})(k).
\end{aligned}
$$

This further implies that for every $(\gamma_1, \gamma_2) \in \Gamma \times \Gamma$, we have

$$
\begin{aligned}
\mathcal{T}(c)(\gamma_1\gamma_2) &= \frac{1}{m_G(C)^2}\int_{C^2} c(g\gamma_1\gamma_2h^{-1})(g)\,\mathrm{d}m_G^{\otimes 2}(g,h)\\
&= \frac{1}{m_G(C)^3}\int_{C^3} (c(g\gamma_1k^{-1})(g) + \pi(\gamma_1)c(k\gamma_2h^{-1})(k))\mathrm{d}m_G^{\otimes 3}(g,h,k)\\
&= \mathcal{T}(c)(\gamma_1) + \pi(\gamma_1)\mathcal{T}(c)(\gamma_2).
\end{aligned}
$$

It follows that the transfer operator $\mathcal{T} : (\mathrm{Z}^1(G, \widehat{\pi}), p_K) \to (\mathrm{Z}^1(\Gamma, \pi), p_S)$ is well-defined and continuous.

(ii) Let $b \in \mathrm{Z}^1(\Gamma, \pi)$. Recall that with the second viewpoint on induction, the induced cocycle $\widehat{b} : G \to \mathcal{H}_{\widehat{\pi}}$ is given by the formula

$$
\forall s, t \in G, \quad \widehat{b}(s)(t) = b(\tau(t^{-1}s, s^{-1}t\Gamma)) - b(\tau(t^{-1}, t\Gamma)).
$$

Set $\gamma^s = \tau(s^{-1}, s\Gamma) \in \Gamma$ for every $s \in G$. It follows that for every $\gamma \in \Gamma$, we have

$$\begin{aligned}
\mathcal{T}(\widehat{b})(\gamma) &= \frac{1}{m_G(C)^2}\int_{C^2} \widehat{b}(g\gamma h^{-1})(g)\,\mathrm{d}m_G^{\otimes 2}(g,h)\\
&= \frac{1}{m_G(C)^2}\int_{C^2} (b(\gamma\gamma^h) - b(\gamma^g))\,\mathrm{d}m_G^{\otimes 2}(g,h)\\
&= \frac{1}{m_G(C)^2}\int_{C^2} (b(\gamma) + \pi(\gamma)b(\gamma^h) - b(\gamma^g))\,\mathrm{d}m_G^{\otimes 2}(g,h).
\end{aligned}$$

Observe that for every $g \in C \subset \mathcal{F}$, we have $\gamma^g = \tau(g^{-1}, g\Gamma) = g^{-1}\sigma(g\Gamma) = e$. Then we have $\mathcal{T}(\widehat{b}) = b$.

(iii) Let $\eta \in \mathcal{H}_{\widehat{\pi}}$. For every $\gamma \in \Gamma$, we have

$$\begin{aligned}
\mathcal{T}(\partial_{\widehat{\pi}}\eta)(\gamma) &= \frac{1}{m_G(C)^2}\int_{C^2} (\partial_{\widehat{\pi}}\eta)(g\gamma h^{-1})(g)\,\mathrm{d}m_G^{\otimes 2}(g,h)\\
&= \frac{1}{m_G(C)^2}\int_{C^2} ((\widehat{\pi}(g\gamma h^{-1})\eta)(g) - \eta(g))\,\mathrm{d}m_G^{\otimes 2}(g,h)\\
&= \frac{1}{m_G(C)^2}\int_{C^2} (\eta(h\gamma^{-1}) - \eta(g))\,\mathrm{d}m_G^{\otimes 2}(g,h)\\
&= \frac{1}{m_G(C)^2}\int_{C^2} (\pi(\gamma)\eta(h) - \eta(g))\,\mathrm{d}m_G^{\otimes 2}(g,h)\\
&= \pi(\gamma)\xi - \xi
\end{aligned}$$

where $\xi = \frac{1}{m_G(C)}\int_C \eta(g)\,\mathrm{d}m_G(g) \in \mathcal{H}_\pi$. Then $\mathcal{T}(\partial_{\widehat{\pi}}\eta) = \partial_\pi\xi$. This finishes the proof of Claim 4.24. □

Let $b \in \mathrm{Z}^1(\Gamma, \pi)$ such that $\widehat{b} \in \mathrm{B}^1(G, \widehat{\pi})$. By combining items (ii) and (iii) in Claim 4.24, we obtain that $b \in \mathrm{B}^1(\Gamma, \pi)$. This proves that the canonical map $\mathrm{H}^1(\Gamma, \pi) \to \mathrm{H}^1(G, \widehat{\pi})$ is injective.

Let $b \in \mathrm{Z}^1(\Gamma, \pi)$ such that $\widehat{b} \in \overline{\mathrm{B}^1(G, \widehat{\pi})}$. By combining items (i), (ii) and (iii) in Claim 4.24, we obtain that $b \in \overline{\mathrm{B}^1(\Gamma, \pi)}$. This proves that the canonical map $\overline{\mathrm{H}}^1(\Gamma, \pi) \to \overline{\mathrm{H}}^1(G, \widehat{\pi})$ is injective. □

CHAPTER 5

Bader–Shalom's normal subgroup theorem

In this chapter, we prove Bader–Shalom's normal subgroup theorem (NST) for irreducible lattices in product groups [**BS04**]. The proof follows Margulis's strategy that consists in proving a "property (T) half" and a "amenability half" and relies on the main results from Chapters 3 and 4.

Introduction

DEFINITION 5.1. Let G be any locally compact group. We say that G is

- *just noncompact* if G is noncompact and for every nontrivial closed normal subgroup $N \lhd G$, the quotient group G/N is compact.
- *topologically simple* if any proper closed normal subgroup is trivial.
- *simple* if any proper normal subgroup is trivial.

Likewise, let Γ be any discrete group. We say that Γ is *just infinite* if Γ is infinite and every nontrivial normal subgroup $N \lhd \Gamma$ has finite index.

Any simple group is topologically simple and any noncompact topologically simple group is just noncompact.

For every $d \geq 1$ and every unital commutative ring R, we define the *projective special linear group*

$$\mathrm{PSL}_d(R) \doteq \mathrm{SL}_d(R)/\mathcal{Z}(\mathrm{SL}_d(R))$$

as the quotient of the special linear group $\mathrm{SL}_d(R)$ by its center $\mathcal{Z}(\mathrm{SL}_d(R))$. For instance, we have

$$\mathrm{PSL}_d(\mathbb{Z}) = \mathrm{SL}_d(\mathbb{Z})/\{\pm 1_d\} \quad \text{and} \quad \mathrm{PSL}_d(\mathbb{R}) = \mathrm{SL}_d(\mathbb{R})/\{\pm 1_d\}.$$

THEOREM 5.2 (Iwasawa). *For every field k and every $d \geq 2$, if $|k| > 3$ or $d > 2$, then $\mathrm{PSL}_d(k)$ is a simple group.*

Theorem 5.2 implies that for every $d \geq 2$, the locally compact group $\mathrm{PSL}_d(\mathbb{R})$ is topologically simple and hence just noncompact. More generally, every simple real Lie group with trivial center is topologically simple.

The following geometric examples of just noncompact locally compact groups are due to Burger–Mozes.

EXAMPLES 5.3 (Burger–Mozes [**BM00a**]). Let $d \geq 3$ be any integer and $\mathsf{T} = (\mathsf{V}, \mathsf{E})$ any d-regular tree. Denote by $\partial\mathsf{T}$ the boundary of T. Denote by $\mathrm{Aut}(\mathsf{T})$ the automorphism group of T endowed with the topology of pointwise convergence. Recall that $\mathrm{Aut}(\mathsf{T})$ is a totally disconnected locally compact group. Any closed subgroup $G < \mathrm{Aut}(\mathsf{T})$ for which the boundary action $G \curvearrowright \partial\mathsf{T}$ is 2-transitive is just noncompact.

The main result of this chapter is the following *normal subgroup theorem* (NST) due to Bader–Shalom.

THEOREM 5.4 (Bader–Shalom [**BS04**]). *For every $i \in \{1, 2\}$, let G_i be a nondiscrete noncompact compactly generated lcsc group. Assume that $(G_1, G_2) \neq (\mathbb{R}, \mathbb{R})$. Let $\Gamma < G_1 \times G_2$ be any finitely generated L^2-integrable weakly uniform irreducible lattice.*

If both G_1 and G_2 are just noncompact, then Γ is just infinite.

Theorem 5.4 applies to all uniform lattices $\Gamma < G_1 \times G_2$ where both G_1 and G_2 are just noncompact compactly generated lcsc groups. Indeed, such uniform lattices $\Gamma < G_1 \times G_2$ are finitely generated by Proposition 1.14, L^2-integrable and weakly uniform by Proposition 2.8.

EXAMPLES 5.5. We give examples of finitely generated L^2-integrable weakly uniform irreducible lattices $\Gamma < G_1 \times G_2$ in products of just noncompact compactly generated lcsc groups to which Theorem 5.4 applies.

(i) For every $d \geq 2$ and every square-free integer $q \geq 2$,
$$\mathrm{PSL}_d(\mathbb{Z}[\sqrt{q}]) < \mathrm{PSL}_d(\mathbb{R}) \times \mathrm{PSL}_d(\mathbb{R})$$
is just infinite

(ii) For every $d \geq 2$ and every prime $p \in \mathcal{P}$,
$$\mathrm{PSL}_d(\mathbb{Z}[p^{-1}]) < \mathrm{PSL}_d(\mathbb{R}) \times \mathrm{PSL}_d(\mathbb{Q}_p)$$
is just infinite

(iii) For every $i \in \{1, 2\}$, let $d_i \geq 3$ and $\mathsf{T}_i = (\mathsf{E}_i, \mathsf{V}_i)$ any d_i-regular tree. Let $G_i < \mathrm{Aut}(\mathsf{T}_i)$ be any closed subgroup for which the boundary action $G_i \curvearrowright \partial\mathsf{T}_i$ is 2-transitive. Any irreducible uniform lattice $\Gamma < G_1 \times G_2$ constructed in [**BM00b**] is just infinite.

Theorem 5.4 extends Margulis's celebrated *normal subgroup theorem* (NST) for irreducible lattices in semisimple algebraic groups.

THEOREM 5.6 (Margulis [**Ma91**, Chapter IV]). *Let G be any higher rank semisimple algebraic group and $\Gamma < G$ any irreducible lattice. For every normal subgroup $N \lhd \Gamma$, either $N \subset \mathscr{Z}(\Gamma)$ and N is finite or $N < \Gamma$ has finite index.*

Let us point out that Bader–Shalom's NST 5.4 generalizes Margulis's NST 5.6 for irreducible lattices in higher rank *nonsimple* semisimple algebraic groups. Bader–Shalom's NST 5.4 also generalizes Burger–Mozes's NST [**BM00b**] for irreducible uniform lattices in product of trees. On the other hand, Margulis's NST 5.6 applies to all lattices $\Gamma < G$ in higher rank simple algebraic groups such as $G = \mathrm{SL}_d(\mathbb{R})$, $d \geq 3$. In that respect, Bader–Shalom's NST 5.4 and Margulis's NST 5.6 are complementary.

The strategy of the proof of Theorem 5.4 follows Margulis's remarkable idea. Let $N \lhd \Gamma$ be any nontrivial normal subgroup. In order to show that N has finite index in Γ, we will prove that the quotient group Γ/N has property (T) (see Theorem 5.8) and is amenable (see Theorem 5.15). Using Proposition 2.27, it will follow that Γ/N is finite. The rest of this chapter is devoted to proving the "property (T) half" and the "amenability half" of Theorem 5.4.

Remark 5.7. Note that the assumption that $(G_1, G_2) \neq (\mathbb{R}, \mathbb{R})$ is necessary. Indeed, regard $\mathbb{Z}^2 < \mathbb{R}^2$ as a uniform irreducible lattice via the *irrational embedding* $\iota : \mathbb{Z}^2 \to \mathbb{R}^2 : (m, n) \mapsto (m - n\sqrt{2}, m + n\sqrt{2})$. Then $\mathbb{Z} \lhd \mathbb{Z}^2$ is an infinite index nontrivial normal subgroup where $\mathbb{Z} \hookrightarrow \mathbb{Z}^2 : m \mapsto (m, m)$.

1. Property (T) half

The main result of this section provides a complete characterization of when factors of irreducible lattices in product groups have property (T). This is the "property (T) half" of Theorem 5.4 which is due to Shalom.

Theorem 5.8 (Shalom [Sh99]). *For every $i \in \{1, 2\}$, let G_i be any compactly generated lcsc group. Set $G = G_1 \times G_2$ and denote by $p_i : G \to G_i$ the canonical factor map. Let $\Gamma < G$ be any finitely generated L^2-integrable weakly uniform irreducible lattice and $N \lhd \Gamma$ any normal subgroup.*

Then Γ/N has property (T) *if and only if the following two conditions are satisfied:*

(i) *For every $i \in \{1, 2\}$, $G_i/\overline{p_i(N)}$ has property* (T).

(ii) *Every continuous homomorphism $\varphi : G \to \mathbb{C}$ that vanishes on N is identically zero.*

It is easy to see that conditions (i) and (ii) are necessary. The heart of the proof will consist in showing that conditions (i) and (ii) are also sufficient. We will use results from Chapter 4.

Before proving Theorem 5.8, we need some preparation. For every $i \in \{1, 2\}$, let G_i be any lcsc group and set $G = G_1 \times G_2$. Let $\Gamma < G$ be any irreducible lattice. Let $p_1 : G \to G_1$ be the canonical factor map.

Definition 5.9. Let $\pi : \Gamma \to \mathcal{U}(\mathcal{H}_\pi)$ be any unitary representation with $\mathcal{H}_\pi$ separable. We say that $\xi \in \mathcal{H}_\pi$ is G_1-*continuous* if for every sequence $(\gamma_n)_n$ in Γ such that $p_1(\gamma_n) \to e$ in G_1, we have $\lim_n \|\pi(\gamma_n)\xi - \xi\| = 0$. We denote by $\mathcal{H}_1 \subset \mathcal{H}_\pi$ the subset of all G_1-continuous elements.

The following result allows us to regard the subset $\mathcal{H}_1 \subset \mathcal{H}_\pi$ of G_1-continuous vectors as the subspace $(\mathcal{H}_{\widehat{\pi}})^{G_2} \subset \mathcal{H}_{\widehat{\pi}}$ of $\widehat{\pi}(G_2)$-invariant vectors in the induced G-space.

Proposition 5.10. *Keep the same notation as above. The following assertions hold.*

(i) *The subset* $\mathcal{H}_1 \subset \mathcal{H}_\pi$ *is a* $\pi(\Gamma)$*-invariant closed susbspace of* $\mathcal{H}_\pi$.

(ii) *The unitary representation* $\pi : \Gamma \to \mathcal{U}(\mathcal{H}_1)$ *extends to a strongly continuous unitary representation* $\pi : G \to \mathcal{U}(\mathcal{H}_1)$ *for which* G_2 *acts trivially.*

(iii) *Denote by* $\widehat{\pi} : G \to \mathcal{U}(\mathcal{H}_{\widehat{\pi}})$ *the induced representation. Then the linear map*

$$W : \mathcal{H}_1 \to (\mathcal{H}_{\widehat{\pi}})^{G_2} : \xi \mapsto (g \mapsto \pi(g^{-1})\xi)$$

is a G*-equivariant unitary operator in the sense that* $W(\pi(g)\xi) = \widehat{\pi}(g)W(\xi)$ *for every* $g \in G$ *and every* $\xi \in \mathcal{H}_1$.

Proof. (i) It is clear that $\mathcal{H}_1 \subset \mathcal{H}_\pi$ is a subspace. Let $\xi \in \mathcal{H}_\pi$ and $(\xi_k)_k$ be any sequence in $\mathcal{H}_1$ such that $\lim_k \|\xi - \xi_k\| = 0$. Let $(\gamma_n)_n$ be any sequence in Γ such that $p_1(\gamma_n) \to e$. Let $\varepsilon > 0$ and choose $k \in \mathbb{N}$ such that $\|\xi - \xi_k\| \leq \varepsilon$. Since $\xi_k \in \mathcal{H}_1$, we have $\lim_n \|\pi(\gamma_n)\xi_k - \xi_k\| = 0$. This implies that $\limsup_n \|\pi(\gamma_n)\xi - \xi\| \leq 2\varepsilon$. Since this is true for every $\varepsilon > 0$, it follows that $\xi \in \mathcal{H}_1$ and hence $\mathcal{H}_1 \subset \mathcal{H}_\pi$ is closed subspace. Let now $\xi \in \mathcal{H}_1$ and $\gamma \in \Gamma$. Let $(\gamma_n)_n$ be any sequence in Γ such that $p_1(\gamma_n) \to e$. Then we have $p_1(\gamma\gamma_n\gamma^{-1}) = p_1(\gamma)p_1(\gamma_n)p_1(\gamma)^{-1} \to e$. Since $\xi \in \mathcal{H}_1$, we have

$$\|\pi(\gamma_n)\,\pi(\gamma)\xi - \pi(\gamma)\xi\| = \|\pi(\gamma\gamma_n\gamma^{-1})\xi - \xi\| \to 0.$$

This shows that $\mathcal{H}_1 \subset \mathcal{H}_\pi$ is $\pi(\Gamma)$-invariant.

(ii) Set $\Lambda \doteq \left\{(m,n) \in \mathbb{N}^2 \mid m \leq n\right\}$. Let $g \in G$ be any element and choose a sequence $(\gamma_n)_n$ in Γ so that $p_1(\gamma_n) \to p_1(g)$. Define the sequence $(\gamma(m,n))_{(m,n)\in\Lambda}$ in Γ by $\gamma(m,n) = \gamma_n^{-1}\gamma_m$ for every $(m,n) \in \Lambda$. Then $p_1(\gamma(m,n)) = p_1(\gamma_n)^{-1}p_1(\gamma_m) \to e$ as $(m,n) \to (+\infty,+\infty)$. This implies that for every $\xi \in \mathcal{H}_1$, we have

$$\|\pi(\gamma_n)\xi - \pi(\gamma_m)\xi\| = \|\xi - \pi(\gamma(m,n))\xi\| \to 0 \quad \text{as} \quad (m,n) \to (+\infty,+\infty)$$

and so the sequence $(\pi(\gamma_n)\xi)_n$ is Cauchy in $\mathcal{H}_1$. Since $\mathcal{H}_1 \subset \mathcal{H}_\pi$ is a closed subspace, there exists $\xi_g \in \mathcal{H}_1$ such that $\lim_n \|\xi_g - \pi(\gamma_n)\xi\| = 0$. It is immediate to check that $\xi_g \in \mathcal{H}_1$ does not depend on the choice of the sequence $(\gamma_n)_n$ for which $p_1(\gamma_n) \to p_1(g)$. Define

$$\rho : G \to \mathcal{U}(\mathcal{H}_1) : g \mapsto (\xi \mapsto \xi_g)\,.$$

We show that $\rho : G \to \mathcal{U}(\mathcal{H}_1)$ is a strongly continuous unitary representation such that $\rho|_\Gamma = \pi$ and $\rho|_{G_2} = 1_{\mathcal{H}_1}$. Firstly, let $g \in G$ and choose a sequence $(\gamma_n)_n$ in Γ so that $p_1(\gamma_n) \to p_1(g)$. For all $\xi, \eta \in \mathcal{H}_1$, we have

$$\langle \rho(g)\xi, \rho(g)\eta\rangle = \langle \xi_g, \eta_g\rangle = \lim_n \langle \pi(\gamma_n)\xi, \eta_g\rangle = \lim_n \langle \xi, \pi(\gamma_n)^*\eta_g\rangle = \langle \xi, \eta\rangle.$$

Then $\rho(g) \in \mathcal{U}(\mathcal{H}_1)$. Next, let $g_1, g_2 \in G$ and choose sequences $(\gamma_{1,n})_n$ and $(\gamma_{2,n})_n$ in Γ so that $p_1(\gamma_{1,n}) \to p_1(g_1)$ and $p_1(\gamma_{2,n}) \to p_1(g_2)$. Then $p_1(\gamma_{1,n}\gamma_{2,n}) \to p_1(g_1g_2)$. Then for all $\xi \in \mathcal{H}_1$, we have

$$\begin{aligned}\|\rho(g_1g_2)\xi - \rho(g_1)\rho(g_2)\xi\| &= \lim_n \|\pi(\gamma_{1,n}\gamma_{2,n})\xi - \rho(g_1)\rho(g_2)\xi\| \\ &= \lim_n \|\pi(\gamma_{2,n})\xi - \pi(\gamma_{1,n})^*\rho(g_1)\,\rho(g_2)\xi\| = 0.\end{aligned}$$

Then $\rho : G \to \mathcal{U}(\mathcal{H}_1)$ is a unitary representation. Let us prove that $\rho : G \to \mathcal{U}(\mathcal{H}_1)$ is continuous at $e \in G$. This will imply that $\rho : G \to \mathcal{U}(\mathcal{H}_1)$ is strongly continuous. Let $\xi \in \mathcal{H}_1$ be any element. Let $(g_k)_k$ be any sequence in G such that $g_k \to e$. By contradiction, assume that $\lim_k \|\rho(g_k)\xi - \xi\| \neq 0$. Up to taking a subsequence, we may assume that $\varepsilon \doteq \inf\{\|\rho(g_k)\xi - \xi\| \mid k \in \mathbb{N}\} > 0$. For every $k \in \mathbb{N}$, choose a sequence $(\gamma_{k,n})_n$ in Γ such that $p_1(\gamma_{k,n}) \to p_1(g_k)$ as $n \to +\infty$. Choose a sequence of open neighborhoods $O_m \subset G_1$ of $e \in G_1$ such that $\bigcap_{m\in\mathbb{N}} O_m = \{e\}$. Since $p_1(g_k) \to e$ as $k \to +\infty$, we can find increasing sequences $(k_m)_m$ and $(n_m)_m$ in $\mathbb{N}$ such that $p_1(g_{k_m}) \in O_m$, $p_1(\gamma_{k_m,n_m}) \in O_m$ and $\|\rho(g_{k_m})\xi - \pi(\gamma_{k_m,n_m})\xi\| < \varepsilon/2$ for every $m \in \mathbb{N}$. Then we have $p_1(\gamma_{k_m,n_m}) \to e$ as $m \to +\infty$ and hence $\lim_m \|\xi - \pi(\gamma_{k_m,n_m})\xi\| = 0$. This further implies that $\limsup_m \|\rho(g_{k_m})\xi - \xi\| \leq \varepsilon/2$, a contradiction. Thus, $\rho : G \to \mathcal{U}(\mathcal{H}_1)$ is a strongly continuous unitary representation. By construction, we have $\rho(\gamma) = \pi(\gamma)$ for every $\gamma \in \Gamma$ and $\rho(g) = 1_{\mathcal{H}_1}$ for every $g \in G_2$.

(iii) We adopt the second viewpoint on induction. Fix a Haar measure m_G on G and a Borel fundamental domain $\mathcal{F} \subset G$ so that $G = \mathcal{F} \cdot \Gamma$ and $0 < m_G(\mathcal{F}) < +\infty$. Recall that we view $\mathcal{H}_{\widehat{\pi}}$ as the Hilbert space of m_G-equivalence classes of all measurable functions $\eta : G \to \mathcal{H}_\pi$ that satisfy

- For m_G-almost every $g \in G$ and every $\gamma \in \Gamma$, $\eta(g\gamma^{-1}) = \pi(\gamma)\eta(g)$.
- $\int_{\mathcal{F}} \|\eta(g)\|^2 \, \mathrm{d}m_G(g) < +\infty$.

The map $W : \mathcal{H}_1 \to (\mathcal{H}_{\widehat{\pi}})^{G_2}$ clearly preserves inner products and is G-equivariant in the sense that $W(\pi(g)\xi) = \widehat{\pi}(g)W(\xi)$ for every $g \in G$ and every $\xi \in \mathcal{H}_1$. It remains to prove that W is surjective. Let $\eta \in (\mathcal{H}_{\widehat{\pi}})^{G_2}$ be any element.

Claim 5.11. Every essential value of η is an element of $\mathcal{H}_1$.

Indeed, let $\xi \in \mathcal{H}_\pi$ be any essential value of η and choose a sequence $(\gamma_n)_{n\in\mathbb{N}}$ in Γ such that $p_1(\gamma_n) \to e$ in G_1. We show that $\lim_n \|\pi(\gamma_n)\xi - \xi\| = 0$. We may find elements $h_n \in G_2$ such that $\gamma_n h_n \to e$ in G. Let $\varepsilon > 0$. Choose a Borel probability measure $\mu \in \mathrm{Prob}(G)$ such that $\mu \sim m_G$ and hence the right translation action $G \curvearrowright (G, \mu)$ is nonsingular. By assumption, the measurable subset $B \doteq \{g \in G \mid \|\eta(g) - \xi\| < \varepsilon\}$ satisfies $\mu(B) > 0$. Since $\gamma_n h_n \to e$ in G, for any $n \in \mathbb{N}$ sufficiently large, we have $\mu(B \cap B \cdot (\gamma_n h_n)^{-1}) > 0$. As an element of $(\mathcal{H}_{\widehat{\pi}})^{G_2}$, the function $\eta : G \to \mathcal{H}_\pi$ is left G_2-invariant (so right G_2-invariant as well since $G_2 \lhd G$ is a normal subgroup) and right Γ-equivariant. Thus for every $n \in \mathbb{N}$ and almost every $g \in G$, we have $\eta(g\gamma_n h_n) = \eta(g\gamma_n) = \pi(\gamma_n^{-1})\eta(g)$. So for any $n \in \mathbb{N}$ sufficiently large, choosing $g \in B \cap B \cdot (\gamma_n h_n)^{-1}$, we obtain

$$\|\xi - \pi(\gamma_n)\xi\| \leq \|\xi - \eta(g)\| + \|\eta(g) - \pi(\gamma_n)\xi\| \leq \|\xi - \eta(g)\| + \|\eta(g\gamma_n h_n) - \xi\| \leq 2\varepsilon.$$

As $\varepsilon > 0$ can be arbitrarily small, this finishes the proof of Claim 5.11.

Using Claim 5.11, we may modify $\eta \in (\mathcal{H}_{\widehat{\pi}})^{G_2}$ on a m_G-null set if necessary and regard $\eta : G \to \mathcal{H}_1$. Then the measurable function $G \to \mathcal{H}_1 : g \mapsto \pi(g)\eta(g)$ is well-defined, it is right G_2-invariant and also right Γ-invariant.

Since the subset $G_2 \cdot \Gamma$ is dense in G, this implies that the above measurable function is right G-invariant and hence m_G-almost everywhere constant. If we denote by $\xi \in \mathcal{H}_1$ its unique essential value, we find that $\xi \in \mathcal{H}_1$ and $\eta = W(\xi)$. □

The next dichotomy result will be one the key ingredients in the proof of Theorem 5.8.

PROPOSITION 5.12 (Shalom [Sh99]). *For every $i \in \{1, 2\}$, let G_i be any compactly generated lcsc group and set $G = G_1 \times G_2$. Let $\pi : G \to \mathcal{U}(\mathcal{H}_\pi)$ be any strongly continuous unitary representation. At least one of the following assertions holds:*

(i) $\overline{\mathrm{H}}^1(G, \pi) = 0$.
(ii) *There exists $i \in \{1, 2\}$ such that $\pi|_{G_i}$ is not ergodic.*

PROOF. For every $i \in \{1, 2\}$, choose a Borel probability measure $\mu_i \in \mathrm{Prob}(G_i)$ as in Terminology 4.6. Set $\mu = \mu_1 \otimes \mu_2 \in \mathrm{Prob}(G)$. Assume that $\pi|_{G_i}$ is ergodic for every $i \in \{1, 2\}$. In order to show that $\overline{\mathrm{H}}^1(G, \pi) = 0$, using Theorem 4.7, it suffices to show that any μ-harmonic 1-cocycle $b \in \mathrm{Har}_\mu(G, \pi)$ is identically zero.

Let $b \in \mathrm{Har}_\mu(G, \pi)$ be any μ-harmonic 1-cocycle. Recall that we have $b(\mu) \doteq \int_G b(g)\, \mathrm{d}\mu(g) = 0$. For every $i \in \{1, 2\}$, set $b(\mu_i) \doteq \int_{G_i} b(g_i)\, \mathrm{d}\mu_i(g_i) \in \mathcal{H}_\pi$ and $\pi(\mu_i) \doteq \int_{G_i} \pi(g_i)\, \mathrm{d}\mu_i(g_i) \in \mathbb{B}(\mathcal{H}_\pi)$. Since $G = G_1 \times G_2$ and $\mu = \mu_1 \otimes \mu_2$, we have

$$\pi(\mu) \doteq \int_G \pi(g)\, \mathrm{d}\mu(g) = \int_{G_1 \times G_2} \pi(g_1)\, \pi(g_2)\, \mathrm{d}(\mu_1 \otimes \mu_2)(g_1, g_2) = \pi(\mu_1)\, \pi(\mu_2).$$

Using the 1-cocycle relation, for every $(g_1, g_2) \in G_1 \times G_2$, we have

$$b(g_1) + \pi(g_1)\, b(g_2) = b(g_1 g_2) = b(g_2 g_1) = b(g_2) + \pi(g_2)\, b(g_1). \tag{5.1}$$

By integrating (5.1) against $\mu = \mu_1 \otimes \mu_2 \in \mathrm{Prob}(G)$ and since $b(\mu) = 0$, we obtain

$$b(\mu_1) + \pi(\mu_1)\, b(\mu_2) = 0 = b(\mu_2) + \pi(\mu_2)\, b(\mu_1).$$

This implies that

$$b(\mu_1) = -\pi(\mu_1)\, b(\mu_2) = -\pi(\mu_1)\, (-\pi(\mu_2)\, b(\mu_1)) = \pi(\mu)\, b(\mu_1).$$

Since π is ergodic, we have $\ker(1 - \pi(\mu)) = \{0\}$ and so $b(\mu_1) = 0$. Likewise, we have $b(\mu_2) = 0$.

Next, for every $(g_1, g_2) \in G_1 \times G_2$, rewriting (5.1) as

$$(1 - \pi(g_1))\, b(g_2) = (1 - \pi(g_2))\, b(g_1). \tag{5.2}$$

By integrating (5.2) against $\mu_1 \in \mathrm{Prob}(G_1)$, for every $g_2 \in G_2$, we obtain

$$(1 - \pi(\mu_1))\, b(g_2) = (1 - \pi(g_2))\, b(\mu_1) = 0.$$

Since $\pi|_{G_1}$ is ergodic, we have $\ker(1 - \pi(\mu_1)) = \{0\}$ and so $b(g_2) = 0$ for every $g_2 \in G_2$. Likewise, we have $b(g_1) = 0$ for every $g_1 \in G_1$. The 1-cocycle relation implies that $b \equiv 0$. □

The next result allows to extend homomorphisms $\varphi : \Gamma \to \mathbb{C}$ to continuous homomorphisms $\overline{\varphi} : G \to \mathbb{C}$.

Proposition 5.13 (Shalom [Sh99]). *For every $i \in \{1, 2\}$, let G_i be any compactly generated lcsc group and set $G = G_1 \times G_2$. Let $\Gamma < G$ be any finitely generated L^2-integrable weakly uniform irreducible lattice. Then any homomorphism $\varphi : \Gamma \to \mathbb{C}$ extends continuously to $\overline{\varphi} : G \to \mathbb{C}$.*

Proof. We can identify the vector space $\mathrm{Hom}(G, \mathbb{C})$ of all continuous homomorpshisms $G \to \mathbb{C}$ with $\mathrm{H}^1(G, 1_G)$ and the vector space $\mathrm{Hom}(\Gamma, \mathbb{C})$ with $\mathrm{H}^1(\Gamma, 1_\Gamma)$. We claim that the restriction map

$$\mathrm{Hom}(G, \mathbb{C}) \to \mathrm{Hom}(\Gamma, \mathbb{C}) : \varphi \mapsto \varphi|_\Gamma$$

is injective. Indeed, let $\varphi \in \mathrm{Hom}(G, \mathbb{C})$ such that $\varphi|_\Gamma = 0$. Set $H = \ker(\varphi)$ and observe that $H \lhd G$ is a closed normal subgroup and $\Gamma < H$. The factor map $G/\Gamma \to G/H : g\Gamma \mapsto gH$ is well-defined and G-equivariant. Since $\Gamma < G$ is a lattice, the group G/H carries a G-invariant Borel probability measure and so G/H is a compact group. Considering the well-defined continuous group homomorphism $G/H \to \mathbb{C} : gH \mapsto \varphi(g)$, we have $\varphi(G) \cong G/H$ and so $\varphi(G)$ is a compact subgroup of $\mathbb{C}$. Since the only compact subgroup of $\mathbb{C}$ is $\{0\}$, we necessarily have $\varphi(G) = 0$ and so $\varphi \equiv 0$. Next, the space $\mathrm{Hom}(\Gamma, \mathbb{C})$ is finite dimensional since Γ is finitely generated. Since $\mathrm{Hom}(G, \mathbb{C}) \to \mathrm{Hom}(\Gamma, \mathbb{C}) : \varphi \mapsto \varphi|_\Gamma$ is injective, $\mathrm{Hom}(G, \mathbb{C})$ is also finite dimensional. Thus, in order to show that the restriction map $\mathrm{Hom}(G, \mathbb{C}) \to \mathrm{Hom}(\Gamma, \mathbb{C})$ is an isomorphism, it suffices to show that there exists an injection $\mathrm{H}^1(\Gamma, 1_\Gamma) \hookrightarrow \mathrm{H}^1(G, 1_G)$.

Denote by $\nu \in \mathrm{Prob}(G/\Gamma)$ the unique G-invariant Borel probability measure and by $\kappa_{G/\Gamma} : G \to \mathcal{U}(\mathrm{L}^2(G/\Gamma, \nu)^0)$ the restriction of the quasi-regular representation $\lambda_{G/\Gamma}$ to the orthogonal complement $\mathrm{L}^2(G/\Gamma, \nu)^0 = \mathrm{L}^2(G/\Gamma, \nu) \ominus \mathbb{C}\mathbf{1}_{G/\Gamma}$. Since $\Gamma < G$ is a weakly uniform lattice, $\kappa_{G/\Gamma}$ does not have almost invariant vectors. Then Proposition 4.4 implies that

$$\mathrm{H}^1(G, \kappa_{G/\Gamma}) = \overline{\mathrm{H}}^1(G, \kappa_{G/\Gamma}).$$

Next, we claim that $\overline{\mathrm{H}}^1(G, \kappa_{G/\Gamma}) = 0$. Indeed, otherwise, applying Proposition 5.12 to $\pi = \kappa_{G/\Gamma}$ and up to permuting the indices, there exists a unit vector $\xi \in \mathrm{L}^2(G/\Gamma, \nu)^0$ such that $\kappa_{G/\Gamma}(g)\xi = \xi$ for all $g \in G_1$. Since $\Gamma < G_1 \times G_2$ is irreducible, the pmp action $G_1 \curvearrowright G/\Gamma$ is ergodic and so the G_1-representation $\kappa_{G/\Gamma}|_{G_1}$ is ergodic. This is a contradiction.

Therefore, we have $\mathrm{H}^1(G, \kappa_{G/\Gamma}) = \overline{\mathrm{H}}^1(G, \kappa_{G/\Gamma}) = 0$. Writing $\lambda_{G/\Gamma} = \kappa_{G/\Gamma} \oplus 1_G$ with respect to the orthogonal sum $\mathrm{L}^2(G/\Gamma, \nu) = \mathrm{L}^2(G/\Gamma, \nu)^0 \oplus \mathbb{C}\mathbf{1}_{G/\Gamma}$, it follows that $\mathrm{H}^1(G, \lambda_{G/\Gamma}) = \mathrm{H}^1(G, 1_G)$. Next, applying Theorem 4.22 to the trivial Γ-representation $\pi = 1_\Gamma$, we have $\widehat{\pi} = \lambda_{G/\Gamma}$ and $\mathrm{H}^1(\Gamma, 1_\Gamma) \hookrightarrow \mathrm{H}^1(G, \lambda_{G/\Gamma})$ is injective, which in turn implies that the map $\mathrm{H}^1(\Gamma, 1_\Gamma) \hookrightarrow \mathrm{H}^1(G, 1_G)$ is injective. $\square$

We are ready to prove Theorem 5.8.

Proof of Theorem 5.8. Assume that Γ/N has property (T). Let $i \in \{1,2\}$. Since $N \lhd \Gamma$ is normal and since $p_i(\Gamma) < G_i$ is dense, $\overline{p_i(N)} \lhd G_i$ is a closed normal subgroup. Moreover, the well-defined group homomorphism $\Gamma/N \to G_i/\overline{p_i(N)}$ has dense range. Therefore, $G_i/\overline{p_i(N)}$ has property (T) by Proposition 2.26. Let now $\varphi : G \to \mathbb{C}$ be any continuous homomorphism such that $\varphi|_N = 0$. Since Γ/N has property (T) and since $\mathbb{C}$ is abelian, the closure of $\varphi(\Gamma/N)$ is compact. Since the only compact subgroup of $\mathbb{C}$ is $\{0\}$, it follows that $\varphi(\Gamma/N) = 0$. The well-defined group homomorphism $\Gamma/N \to \mathbb{C} : \gamma N \mapsto \varphi(\gamma)$ must be identically zero. Thus, $\varphi|_\Gamma = 0$. As explained in the proof of Proposition 5.13, this further implies that $\varphi \equiv 0$.

Conversely, assume that assumptions (i) and (ii) are satisfied. By contradiction, assume that Γ/N does not have property (T). Choose a symmetric finitely supported probability measure $\mu \in \mathrm{Prob}(\Gamma)$ whose support $\mathrm{supp}(\mu)$ generates Γ. Denote by $\overline{\mu} \in \mathrm{Prob}(\Gamma/N)$ the pushforward measure under the factor map $\Gamma \to \Gamma/N$. Then $\overline{\mu} \in \mathrm{Prob}(\Gamma/N)$ satisfies the assumption of Terminology 4.6 for the factor group Γ/N. By Theorems 4.7 and 4.10, there exist a unitary representation $\pi : \Gamma/N \to \mathcal{U}(\mathcal{H}_\pi)$ and a nonzero $\overline{\mu}$-harmonic cocycle $b \in \mathrm{Har}_{\overline{\mu}}(\Gamma/N, \pi)$. Replacing $\mathcal{H}_\pi$ with the closure of the linear span of $b(\Gamma/N)$, we may assume that $\mathcal{H}_\pi$ is separable. We regard π as a Γ-unitary representation such that $\pi|_N = 1_{\mathcal{H}_\pi}$ and $b \in \mathrm{Har}_\mu(\Gamma, \pi)$ as a μ-harmonic cocycle for $\pi : \Gamma \to \mathcal{U}(\mathcal{H}_\pi)$ such that $b|_N = 0$.

Step 1: One may assume that π is ergodic.

Consider the orthogonal decomposition $\mathcal{H}_\pi = (\mathcal{H}_\pi \ominus (\mathcal{H}_\pi)^\Gamma) \oplus (\mathcal{H}_\pi)^\Gamma$ into $\pi(\Gamma)$-invariant subspaces. We claim that $p_{(\mathcal{H}_\pi)^\Gamma} \circ b = 0$. Indeed, otherwise there exists a nonzero vector $\xi \in (\mathcal{H}_\pi)^\Gamma$, for which the homomorphism $\varphi : \Gamma \to \mathbb{C} : \gamma \mapsto \langle p_{(\mathcal{H}_\pi)^\Gamma}(b(\gamma)), \xi\rangle$ is nontrivial and satisfies $\varphi|_N = 0$. Using Proposition 5.13, $\varphi : \Gamma \to \mathbb{C}$ extends to a nontrivial homomorphism $\overline{\varphi} : G \to \mathbb{C}$. This contradicts assumption (ii). Thus, up to replacing $(\pi, \mathcal{H}_\pi)$ with $(\pi, \mathcal{H}_\pi \ominus (\mathcal{H}_\pi)^\Gamma)$ and b with $p_{\mathcal{H}_\pi \ominus (\mathcal{H}_\pi)^\Gamma} \circ b$, we may assume that π is ergodic and $b \neq 0$. Then the induced representation $\widehat{\pi} : G \to \mathcal{U}(\mathcal{H}_{\widehat{\pi}})$ is ergodic by Proposition 2.13.

Step 2: One may assume that π extends to a strongly continuous G-unitary representation.

Using Proposition 5.10(i), for every $j \in \{1,2\}$, denote by $\mathcal{H}_j \subset \mathcal{H}_\pi$ the $\pi(\Gamma)$-invariant closed subspace of G_j-continuous elements. Using Proposition 5.10(ii), the unitary representation $\pi : \Gamma \to \mathcal{U}(\mathcal{H}_1)$ (resp. $\pi : \Gamma \to \mathcal{U}(\mathcal{H}_2)$) extends to a strongly continuous unitary representation $\pi : G \to \mathcal{U}(\mathcal{H}_1)$ (resp. $\pi : G \to \mathcal{U}(\mathcal{H}_2)$) for which G_2 (resp. G_1) acts trivially. Moreover, Proposition 5.10(iii) shows that $W_1 : \mathcal{H}_1 \to (\mathcal{H}_{\widehat{\pi}})^{G_2} : \xi \mapsto (g \mapsto \pi(g^{-1})\xi)$ (resp. $W_2 : \mathcal{H}_2 \to (\mathcal{H}_{\widehat{\pi}})^{G_1} : \xi \mapsto (g \mapsto \pi(g^{-1})\xi)$) is a G-equivariant unitary operator.

We claim that $\mathcal{H}_1$ and $\mathcal{H}_2$ are orthogonal in $\mathcal{H}_\pi$. Indeed, for every $\xi_1 \in \mathcal{H}_1$, every $\xi_2 \in \mathcal{H}_2$ and every $g \in G$, we have

$$\langle \xi_1, \xi_2 \rangle = \langle \pi(g^{-1})\xi_1, \pi(g^{-1})\xi_2 \rangle.$$

This implies that

$$\langle W_1(\xi_1), W_2(\xi_2)\rangle_{\mathcal{H}_{\widehat{\pi}}} = \int_{\mathcal{F}} \langle \pi(g^{-1})\xi_1, \pi(g^{-1})\xi_2 \rangle \, \mathrm{d}m_G(g) = m_G(\mathcal{F}) \cdot \langle \xi_1, \xi_2 \rangle.$$

Since $\widehat{\pi}$ is ergodic and $G = G_1 \times G_2$, the closed subspaces $(\mathcal{H}_{\widehat{\pi}})^{G_1}$ and $(\mathcal{H}_{\widehat{\pi}})^{G_2}$ are othogonal in $\mathcal{H}_{\widehat{\pi}}$. Thus, $\langle \xi_1, \xi_2 \rangle = \frac{1}{m_G(\mathcal{F})} \langle W_1(\xi_1), W_2(\xi_2)\rangle_{\mathcal{H}_{\widehat{\pi}}} = 0$ and so $\mathcal{H}_1$ and $\mathcal{H}_2$ are orthogonal in $\mathcal{H}_\pi$.

Denote by $\mathcal{K} \subset \mathcal{H}_\pi$ the orthogonal complement of $\mathcal{H}_1 \oplus \mathcal{H}_2$ in $\mathcal{H}_\pi$ and observe that $\mathcal{K}$ is $\pi(\Gamma)$-invariant. We have the following orthogonal decomposition $\mathcal{H}_\pi = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{K}$ into $\pi(\Gamma)$-invariant subspaces. This further yields the following orthogonal decomposition $\mathcal{H}_{\widehat{\pi}} = \widehat{\mathcal{H}}_1 \oplus \widehat{\mathcal{H}}_2 \oplus \widehat{\mathcal{K}}$ into $\widehat{\pi}(G)$-invariant subspaces. Note that Proposition 5.10(iii) implies that $(\mathcal{H}_{\widehat{\pi}})^{G_2} = (\widehat{\mathcal{H}}_1)^{G_2} \subset \widehat{\mathcal{H}}_1$ and $(\mathcal{H}_{\widehat{\pi}})^{G_1} = (\widehat{\mathcal{H}}_2)^{G_1} \subset \widehat{\mathcal{H}}_2$. Then Proposition 5.12 applied to $(\widehat{\pi}, \widehat{\mathcal{K}})$ implies that $\overline{\mathrm{H}}^1(\widehat{\pi}, \widehat{\mathcal{K}}) = 0$. Then Theorem 4.22 implies that $p_{\mathcal{K}} \circ b = 0$. Therefore, up to replacing $(\pi, \mathcal{H}_\pi)$ with $(\pi, \mathcal{H}_1 \oplus \mathcal{H}_2)$ and b with $p_{\mathcal{H}_1 \oplus \mathcal{H}_2} \circ b$, we may assume that $\mathcal{H}_\pi = \mathcal{H}_1 \oplus \mathcal{H}_2$ and $b \neq 0$. Up to permuting the indices and replacing $(\pi, \mathcal{H}_\pi)$ with $(\pi, \mathcal{H}_1)$ and b with $p_{\mathcal{H}_1} \circ b$, we may further assume that $\mathcal{H}_\pi = \mathcal{H}_1$ and $b \neq 0$.

Step 3 : Extending Γ-cocycles to G-cocycles for π.

We may now regard the ergodic unitary representation $\pi : \Gamma \to \mathcal{U}(\mathcal{H}_\pi)$ as an ergodic strongly continuous unitary representation $\pi : G \to \mathcal{U}(\mathcal{H}_\pi)$ such that $\pi|_{G_2} = 1_{\mathcal{H}_\pi}$. By Example 2.12(iii), $\widehat{\pi}$ is unitarily equivalent to $\pi \otimes \lambda_{G/\Gamma}$.

Observe that $(\mathcal{H}_{\widehat{\pi}})^{G_1} \subset \widehat{\mathcal{H}}_2 = 0$ hence $\widehat{\pi}$ is G_1-ergodic. Set

$$c = p_{(\mathcal{H}_{\widehat{\pi}})^{G_2}} \circ \widehat{b} \in \mathrm{Z}^1(G, \widehat{\pi}, (\mathcal{H}_{\widehat{\pi}})^{G_2}).$$

Then Proposition 5.12 applied to $(\widehat{\pi}, \mathcal{H}_{\widehat{\pi}} \ominus (\mathcal{H}_{\widehat{\pi}})^{G_2})$ implies that

$$\mathfrak{d} \doteq \widehat{b} - c \in \overline{\mathrm{B}^1(G, \widehat{\pi}, \mathcal{H}_{\widehat{\pi}} \ominus (\mathcal{H}_{\widehat{\pi}})^{G_2})}.$$

Recall that the linear mapping $W : \mathcal{H}_\pi \to (\mathcal{H}_{\widehat{\pi}})^{G_2} : \xi \mapsto (h \mapsto \pi(h^{-1})\xi)$ is a unitary operator such that $W\pi(g)W^* = \widehat{\pi}(g)|_{(\mathcal{H}_{\widehat{\pi}})^{G_2}}$ for every $g \in G$. Define the mapping $b_c : G \to \mathcal{H}_\pi : g \mapsto W^* c(g)$ and observe that $b_c \in \mathrm{Z}^1(G, \pi)$. Consider the transfer operator $\mathcal{T} : \mathrm{Z}^1(G, \widehat{\pi}) \to \mathrm{Z}^1(\Gamma, \pi)$ associated with a relatively compact subset $C \subset \mathcal{F}$ as in Claim 4.24.

Claim 5.14. Regarding $b_c \in \mathrm{Z}^1(\Gamma, \pi)$, we have $b - b_c \in \overline{\mathrm{B}^1(\Gamma, \pi)}$.

Indeed, by Claim 4.24, we have $\mathcal{T}(\widehat{b}) = b$ and $\mathcal{T}(\mathfrak{d}) \in \overline{\mathrm{B}^1(\Gamma, \pi)}$. Next, a simple calculation shows that for every $\gamma \in \Gamma$, we have

$$\mathcal{T}(c)(\gamma) = \frac{1}{m_G(C)^2} \int_{C^2} c(g\gamma h^{-1})(g) \, \mathrm{d}m_G^{\otimes 2}(g, h)$$

$$
\begin{aligned}
&= \frac{1}{m_G(C)^2}\int_{C^2} \pi(g^{-1})\, b_c(g\gamma h^{-1})\, \mathrm{d}m_G^{\otimes 2}(g,h) \\
&= \frac{1}{m_G(C)^2}\int_{C^2} (-b_c(g^{-1}) + b_c(\gamma h^{-1}))\, \mathrm{d}m_G^{\otimes 2}(g,h) \\
&= \frac{1}{m_G(C)^2}\int_{C^2} (-b_c(g^{-1}) + b_c(\gamma) + \pi(\gamma) b_c(h^{-1}))\, \mathrm{d}m_G^{\otimes 2}(g,h) \\
&= b_c(\gamma) + \pi(\gamma)\xi - \xi
\end{aligned}
$$

where $\xi = \frac{1}{m_G(C)}\int_C b_c(g^{-1})\, \mathrm{d}m_G(g) \in \mathcal{H}_\pi$. This implies that $\mathcal{T}(c) = b_c + \partial_\pi \xi$. Therefore, we obtain

$$b - b_c = \mathcal{T}(\widehat{b} - c) + \partial_\pi\xi = \mathcal{T}(\mathfrak{d}) + \partial_\pi\xi \in \overline{\mathrm{B}^1(\Gamma, \pi)}.$$

This finishes the proof of Claim 5.14.

Since $b|_N = 0$ and $\pi|_N = 1_{\mathcal{H}_\pi}$, Claim 5.14 implies that $b_c|_N = 0$. Moreover, for every $g = (g_1, g_2) \in G_1 \times G_2 = G$, we have

$$
\begin{aligned}
b_c(g_1) + \pi(g_1) b_c(g_2) &= b_c(g_1 g_2) \\
&= b_c(g_2 g_1) = b_c(g_2) + \pi(g_2) b_c(g_1) \\
&= b_c(g_2) + b_c(g_1).
\end{aligned}
$$

This implies that $b_c(g_2) \in \mathcal{H}_\pi$ is $\pi(G_1)$-invariant and so $b_c(g_2) = 0$ for every $g_2 \in G_2$. This further implies that $b_c|_{G_2} = 0$ and hence $b_c|_{\overline{p_1(N)}} = 0$.

We may now regard $\pi : G_1/\overline{p_1(N)} \to \mathcal{U}(\mathcal{H}_\pi)$ as a strongly continuous unitary representation of the factor group $G_1/\overline{p_1(N)}$. Moreover, we may regard $b_c \in \mathrm{Z}^1(G_1/\overline{p_1(N)}, \pi)$ as a continuous cocycle for the strongly continuous unitary representation $\pi : G_1/\overline{p_1(N)} \to \mathcal{U}(\mathcal{H}_\pi)$. Since $G_1/\overline{p_1(N)}$ has property (T), we have $b_c \in \mathrm{B}^1(G_1/\overline{p_1(N)}, \pi)$ and hence $b_c : G \to \mathcal{H}_\pi$ is bounded. Finally, Lemma 4.3 and Claim 5.14 imply that $b \in \overline{\mathrm{B}^1(\Gamma, \pi)}$. Since by assumption $b \in \mathrm{Har}_\mu(\Gamma, \pi)$, it follows that $b \equiv 0$ by Theorem 4.7. This is a contradiction. Therefore, Γ/N has property (T). This finishes the proof of Theorem 5.8. □

2. Amenability half

The main result of this section provides a complete characterization of when factors of irreducible lattices in product groups are amenable. This is the "amenability half" of Theorem 5.4 which is due to Bader–Shalom.

THEOREM 5.15 (Bader–Shalom [**BS04**]). *For every $i \in \{1, 2\}$, let G_i be any lcsc group. Set $G = G_1 \times G_2$ and denote by $p_i : G \to G_i$ the canonical factor map. Let $\Gamma < G$ be any irreducible lattice and $N \lhd \Gamma$ any normal subgroup.*

Then Γ/N is amenable if and only if for every $i \in \{1, 2\}$, $G_i/\overline{p_i(N)}$ is amenable.

We will use results from Chapter 3. Before proving Theorem 5.15, we need some preparation. For every $i \in \{1,2\}$, let G_i be any lcsc group and set $G = G_1 \times G_2$. For every $i \in \{1,2\}$, choose an admissible Borel probability measure $\mu_i \in \mathrm{Prob}(G_i)$ and denote by (B_i, ν_{B_i}) the (G_i,μ_i)-Poisson boundary. The next proposition describes any ergodic (G,μ)-space in terms of a canonical relatively measure preserving extension. For every $i \in \{1,2\}$, denote by $j \in \{1,2\}$ the unique element so that $\{1,2\} = \{i,j\}$.

Proposition 5.16 (Bader–Shalom [**BS04**]). *Let (Y,η) be any ergodic (G,μ)-space. The following assertions hold:*

(i) *For every $i \in \{1,2\}$, (Y,η) is a (G_i,μ_i)-space.*
(ii) *For every $i \in \{1,2\}$, the G_i-equivariant measurable factor map $\pi_i : (Y,\eta) \to (Y_i,\eta_i)$ arising from the inclusion $\mathrm{L}^\infty(Y)^{G_j} \subset \mathrm{L}^\infty(Y)$ is relatively measure preserving.*
(iii) *We have that $\pi_1 \otimes \pi_2 : (Y,\eta) \to (Y_1 \times Y_2, \eta_1 \otimes \eta_2)$ is a relatively measure preserving G-equivariant measurable factor map.*

Proof. (i) Let $i \in \{1,2\}$. Set $\zeta_i = \mu_i * \eta \in \mathrm{Prob}(Y)$ and observe that $\zeta_i \prec \eta$ (by Lemma 3.6). Since $G = G_1 \times G_2$ and $\mu = \mu_1 \otimes \mu_2$ and since $\mu * \eta = \eta$, we have

$$\mu * \zeta_i = \mu * \mu_i * \eta = \mu_i * \mu * \eta = \mu_i * \eta = \zeta_i.$$

Since (Y,η) is an ergodic (G,μ)-space, Proposition 3.8(ii) implies that $\zeta_i = \eta$. This shows that (Y,η) is a (G_i,μ_i)-space.

(ii) Observe that $\mathrm{L}^\infty(Y)^{G_j} \subset \mathrm{L}^\infty(Y)$ is a G_i-invariant von Neumann subalgebra. Denote by $\pi_i : (Y,\eta) \to (Y_i,\eta_i)$ the G_i-equivariant measurable factor map such that $\mathrm{L}^\infty(Y_i) = \mathrm{L}^\infty(Y)^{G_j}$. We have $\eta_i = \eta|_{\mathrm{L}^\infty(Y)^{G_j}}$. Since (Y,η) is a (G_i,μ_i)-space, (Y_i,η_i) is also a (G_i,μ_i)-space. Denote by $\mathrm{E}_i : \mathrm{L}^\infty(Y) \to \mathrm{L}^\infty(Y)^{G_j}$ the unique conditional expectation such that $\eta \circ \mathrm{E}_i = \eta$. By Proposition 3.8, we have

$$\forall f \in \mathrm{L}^\infty(Y), \quad T_{\mu_j}(f) = \int_{G_j} \sigma_{g_j}^{-1}(f)\, \mathrm{d}\mu_j(g_j)$$
$$\mathrm{E}_i(f) = \lim_{n \to \mathcal{U}} \frac{1}{n} \sum_{k=1}^n (T_{\mu_j})^{\circ k}(f).$$

Since G_1 and G_2 commute in G and since the action $G \curvearrowright \mathrm{L}^\infty(Y)$ is weak*-continuous, $T_{\mu_j} : \mathrm{L}^\infty(Y) \to \mathrm{L}^\infty(Y)$ is G_i-equivariant. This further implies that $\mathrm{E}_i : \mathrm{L}^\infty(Y) \to \mathrm{L}^\infty(Y)^{G_j}$ is G_i-equivariant. Thus, $\pi_i : (Y,\eta) \to (Y_i,\eta_i)$ is relatively measure preserving.

(iii) Set $A = \mathrm{L}^\infty(Y)$ and $A_i = \mathrm{L}^\infty(Y)^{G_j} = \mathrm{L}^\infty(Y_i)$ for every $i \in \{1,2\}$. We have $\eta_i = \eta|_{A_i}$ and $\eta \circ \mathrm{E}_i = \eta$ for every $i \in \{1,2\}$. Denote by $\sigma : G \curvearrowright A$ the weak*-continuous action. Since $\mathrm{E}_1 : A \to A^{G_2}$ is G_1-equivariant and since $A^G = \mathbb{C}\mathbf{1}_Y$, it follows that $\mathrm{E}_1|_{A^{G_1}} = \eta(\,\cdot\,)\,\mathbf{1}_Y$. Then for every $f_1 \in A_1$ and every $f_2 \in A_2$, we have

$$\eta(f_1 \cdot f_2) = \eta(\mathrm{E}_1(f_1 \cdot f_2)) = \eta(\mathrm{E}_1(f_1) \cdot f_2) = \eta(f_1)\,\eta(f_2).$$

This shows that A_1 and A_2 are η-independent in A and so we write $A_1 \overline{\otimes} A_2 = A_1 \vee A_2 \subset A$ for the G-invariant von Neumann subalgebra generated by A_1 and A_2. Denote by $\mathrm{E} : A \to A_1 \overline{\otimes} A_2$ the unique conditional expectation such that $\eta \circ \mathrm{E} = \eta$. Let $i \in \{1, 2\}$. Since $\mathrm{E}_i \circ \mathrm{E} : A \to A_i$ is a conditional expectation such that $\eta \circ \mathrm{E}_i \circ \mathrm{E} = \eta$, we have $\mathrm{E}_i \circ \mathrm{E} = \mathrm{E}_i$. Let $g_i \in G_i$ be any element. We have

$$\begin{aligned}
\eta \circ \sigma_{g_i} \circ \mathrm{E} \circ \sigma_{g_i}^{-1} &= \eta \circ \mathrm{E}_i \circ \sigma_{g_i} \circ \mathrm{E} \circ \sigma_{g_i}^{-1} \\
&= \eta \circ \sigma_{g_i} \circ \mathrm{E}_i \circ \mathrm{E} \circ \sigma_{g_i}^{-1} \quad (\text{since } \mathrm{E}_i \text{ is } G_i\text{-equivariant}) \\
&= \eta \circ \sigma_{g_i} \circ \mathrm{E}_i \circ \sigma_{g_i}^{-1} \\
&= \eta \circ \mathrm{E}_i \quad (\text{since } \mathrm{E}_i \text{ is } G_i\text{-equivariant}) \\
&= \eta.
\end{aligned}$$

Since $\sigma_{g_i} \circ \mathrm{E} \circ \sigma_{g_i}^{-1} : A \to A_i$ is a conditional expectation such that $\eta \circ \sigma_{g_i} \circ \mathrm{E} \circ \sigma_{g_i}^{-1} = \eta$, we have $\sigma_{g_i} \circ \mathrm{E} \circ \sigma_{g_i}^{-1} = \mathrm{E}$. This shows that $\mathrm{E} : A \to A_1 \overline{\otimes} A_2$ is G_i-equivariant. Likewise, $\mathrm{E} : A \to A_1 \overline{\otimes} A_2$ is G_j-equivariant. Altogether, $\mathrm{E} : A \to A_1 \overline{\otimes} A_2$ is G-equivariant.

Observe that the inclusion $A_1 \overline{\otimes} A_2 \subset A$ corresponds to the G-equivariant measurable factor map $\pi_1 \otimes \pi_2 : (Y, \eta) \to (Y_1 \times Y_2, \eta_1 \otimes \eta_2)$. Since $\mathrm{E} : A \to A_1 \overline{\otimes} A_2$ is G-equivariant, $\pi_1 \otimes \pi_2 : (Y, \eta) \to (Y_1 \times Y_2, \eta_1 \otimes \eta_2)$ is relatively measure preserving. □

The following corollary describes all possible (G, μ)-boundaries in terms of (G_i, μ_i)-boundaries.

Corollary 5.17 (Bader–Shalom [BS04]). *Let (C, ν_C) be any (G, μ)-boundary. For every $i \in \{1, 2\}$, there exists a unique (G_i, μ_i)-boundary (C_i, ν_{C_i}) such that $(C, \nu_C) \cong (C_1 \times C_2, \nu_{C_1} \otimes \nu_{C_2})$ as (G, μ)-spaces.*

In particular, we have $(B, \nu_B) \cong (B_1 \times B_2, \nu_{B_1} \otimes \nu_{B_2})$ as (G, μ)-spaces.

Proof. By Proposition 3.22, we have $\mathrm{L}^\infty(B_i) = \mathrm{L}^\infty(B)^{G_j}$. Let (C, ν_C) be any (G, μ)-boundary and denote by $\pi_C : (B, \nu_B) \to (C, \nu_C)$ the essentially unique G-equivariant measurable factor map. By Proposition 5.16, for every $i \in \{1, 2\}$, denote by $\pi_i : (C, \nu_C) \to (C_i, \nu_{C_i})$ the relatively measure preserving G_i-equivariant measurable factor map arising from $\mathrm{L}^\infty(C_i) = \mathrm{L}^\infty(C)^{G_j}$. Since $\mathrm{L}^\infty(C_i) = \mathrm{L}^\infty(C)^{G_j} \subset \mathrm{L}^\infty(B)^{G_j} = \mathrm{L}^\infty(B_i)$, (C_i, ν_{C_i}) is a (G_i, μ_i)-boundary. Then Proposition 5.16(iii) implies that $\pi_1 \otimes \pi_2 : (C, \nu_C) \to (C_1 \times C_2, \nu_{C_1} \otimes \nu_{C_2})$ is a relatively measure preserving G-equivariant measurable factor map. Then Corollary 3.27 implies that $\pi_1 \otimes \pi_2 : (C, \nu_C) \to (C_1 \times C_2, \nu_{C_1} \otimes \nu_{C_2})$ is an isomorphism and so $(C, \nu_C) \cong (C_1 \times C_2, \nu_{C_1} \otimes \nu_{C_2})$ as (G, μ)-spaces.

Applying the above reasoning to $(C, \nu_C) = (B, \nu_B)$, we have $(B, \nu_B) \cong (B_1 \times B_2, \nu_{B_1} \otimes \nu_{B_2})$ as (G, μ)-spaces. □

Let now $\Gamma < G$ be any irreducible lattice. The key result to prove Theorem 5.15 is the following *factor theorem* that describes all possible Γ-factors of the (G, μ)-Poisson boundary (B, ν_B).

THEOREM 5.18 (Bader–Shalom [BS04]). *Let $\Gamma \curvearrowright (Z,\zeta)$ be any nonsingular action so that there exists a Γ-equivariant measurable factor map $\pi : (B,\nu_B) \to (Z,\zeta)$. Then there exists a (G,μ)-boundary (C,ν_C) and a Γ-equivariant measurable isomorphism $\varphi : (Z,\zeta) \to (C,\nu_C)$ such that $\varphi \circ \pi = \pi_C$ almost everywhere.*

We say that a pmp action $G \curvearrowright (X,\nu)$ is *irreducible* if for every $i \in \{1,2\}$, the restriction $G_i \curvearrowright (X,\nu)$ is ergodic. Before proving Theorem 5.18, we prove the following general *intermediate factor theorem* that describes all possible intermediate G-factors associated with irreducible pmp actions $G \curvearrowright (X,\nu)$.

THEOREM 5.19 (Bader–Shalom [BS04]). *Let $G \curvearrowright (X,\nu)$ be any irreducible pmp action. Let (Y,η) be any (G,μ)-space so that there exist G-equivariant measurable factor maps*

$$(B \times X, \nu_B \otimes \nu) \xrightarrow{\Psi} (Y,\eta) \xrightarrow{\rho} (X,\nu)$$

such that $\rho \circ \Psi = p_X : B \times X \to X$.

Then there exist a (G,μ)-boundary (C,ν_C) with its essentially unique G-equivariant measurable factor map $\pi_C : (B,\nu_B) \to (C,\nu_C)$ and a G-equivariant measurable isomorphism $\Phi : (Y,\eta) \to (C \times X, \nu_C \otimes \nu)$ such that $\Phi \circ \Psi = \pi_C \otimes \mathrm{id}_X$ and $p_X \circ \Phi = \rho$ almost everywhere.

PROOF. Using Proposition 3.3, we may and will assume that all G-spaces considered in the proof are compact metrizable G-spaces. The G-equivariant measurable factor maps

$$(B \times X, \nu_B \otimes \nu) \xrightarrow{\Psi} (Y,\eta) \xrightarrow{\rho} (X,\nu)$$

such that $\rho \circ \Psi = p_X : B \times X \to X$ give rise to the following inclusions of G-invariant von Neumann subalgebras

$$\mathrm{L}^\infty(X) \subset \mathrm{L}^\infty(Y) \subset \mathrm{L}^\infty(B \times X)$$

such that $\nu = \eta|_{\mathrm{L}^\infty(X)}$ and $\eta = (\nu_B \otimes \nu)|_{\mathrm{L}^\infty(Y)}$ and the inclusion

$$\mathbb{C}\mathbf{1}_B \mathbin{\overline{\otimes}} \mathrm{L}^\infty(X) = \mathrm{L}^\infty(X) \subset \mathrm{L}^\infty(B \times X) = \mathrm{L}^\infty(B) \mathbin{\overline{\otimes}} \mathrm{L}^\infty(X)$$

is the diagonal inclusion.

Since $G \curvearrowright (X,\nu)$ is irreducible, we have $\mathrm{L}^\infty(X)^{G_1} = \mathrm{L}^\infty(X)^{G_2} = \mathbb{C}\mathbf{1}_X$. By Corollary 5.17, we have $(B,\nu_B) \cong (B_1 \times B_2, \nu_{B_1} \otimes \nu_{B_2})$ as (G,μ)-spaces. Moreover, Corollary 3.29 implies that $\mathrm{L}^\infty(B \times X)^{G_1} = \mathrm{L}^\infty(B_2)$ and $\mathrm{L}^\infty(B \times X)^{G_2} = \mathrm{L}^\infty(B_1)$. Following Proposition 5.16, denote by $\varphi : (Y,\eta) \to (C,\nu_C)$ the relatively measure preserving G-equivariant measurable factor map corresponding to the inclusion $\mathrm{L}^\infty(C) = \mathrm{L}^\infty(Y)^{G_1} \mathbin{\overline{\otimes}} \mathrm{L}^\infty(Y)^{G_2} \subset \mathrm{L}^\infty(Y)$ where $\nu_C = \eta|_{\mathrm{L}^\infty(C)}$. Since

$$\mathrm{L}^\infty(C) = \mathrm{L}^\infty(Y)^{G_1} \mathbin{\overline{\otimes}} \mathrm{L}^\infty(Y)^{G_2} \subset \mathrm{L}^\infty(B \times X)^{G_1} \mathbin{\overline{\otimes}} \mathrm{L}^\infty(B \times X)^{G_2} = \mathrm{L}^\infty(B)$$

and $\nu_C = \nu_B|_{\mathrm{L}^\infty(C)}$, it follows that (C,ν_C) is a (G,μ)-boundary. Denote by $\pi_C : (B,\nu_B) \to (C,\nu_C)$ the essentially unique G-equivariant measurable factor map. We obtain the following commutative diagram of G-equivariant measurable factor maps:

$$\begin{tikzcd}
(B \times X, \nu_B \otimes \nu) \arrow[r, "\Psi"] \arrow[d, "p_B"] & (Y, \eta) \arrow[r, "\rho"] \arrow[d, "\varphi"] & (X, \nu) \arrow[d] \\
(B, \nu_B) \arrow[r, "\pi_C"] & (C, \nu_C) \arrow[r] & \{*\}
\end{tikzcd}$$

where the vertical arrows are relatively measure preserving G-equivariant measurable factor maps. In particular, we obtain the following inclusions of G-invariant von Neumann subalgebras

$$\mathrm{L}^\infty(C) \mathbin{\overline{\otimes}} \mathrm{L}^\infty(X) \subset \mathrm{L}^\infty(Y) \subset \mathrm{L}^\infty(B) \mathbin{\overline{\otimes}} \mathrm{L}^\infty(X)$$

such that $\nu_C \otimes \nu = \eta|_{\mathrm{L}^\infty(C) \overline{\otimes} \mathrm{L}^\infty(X)}$ and $\eta = (\nu_B \otimes \nu)|_{\mathrm{L}^\infty(Y)}$. We obtain that $\varphi \otimes \rho : (Y, \eta) \to (C \times X, \nu_C \otimes \nu)$ is a G-equivariant measurable factor map. It remains to prove that $\Psi : (B \times X, \nu_B \otimes \nu) \to (Y, \eta)$ factors through $\overline{\Psi} : (C \times X, \nu_C \otimes \nu) \to (Y, \eta)$ and that $(\varphi \otimes \rho) \circ \overline{\Psi} = \mathrm{id}_{C \times X}$ almost everywhere.

Using Theorem 3.20 and the naturality of limit measures as in Corollary 3.25, for $\mathbf{P}$-almost every $\omega \in \Omega$, we obtain the following commutative diagram of measurable factor maps:

$$\begin{tikzcd}
(B \times X, \delta_{\overline{\omega}} \otimes \nu) \arrow[r, "\Psi"] \arrow[d, "p_B"] & (Y, \eta_\omega) \arrow[r, "\rho"] \arrow[d, "\varphi"] & (X, \nu) \arrow[d] \\
(B, \delta_{\overline{\omega}}) \arrow[r, "\pi_C"] & (C, \delta_{\pi_C(\overline{\omega})}) \arrow[r] & \{*\}
\end{tikzcd}$$

Denote by $\mathrm{E} : \mathrm{L}^\infty(Y) \to \mathrm{L}^\infty(C)$ the unique conditional expectation such that $\nu_C \circ \mathrm{E} = \eta$ and consider the restriction $\mathrm{E}\,|_{\mathrm{C}(Y)} : \mathrm{C}(Y) \to \mathrm{L}^\infty(C)$. By duality, we obtain the G-equivariant measurable map $\beta : C \to \mathrm{Prob}(Y)$ such that $\eta = \int_C \beta(c)\, \mathrm{d}\nu_C(c)$. Using Corollary 3.26, for $\mathbf{P}$-almost every $\omega \in \Omega$, we have $\beta(\pi_C(\overline{\omega})) = \eta_\omega$. Thus, we may consider the well-defined G-equivariant measurable map $C \to \mathrm{Prob}(Y) : \pi_C(\overline{\omega}) \mapsto \eta_\omega \doteq \eta_{\pi_C(\overline{\omega})}$. The above commutative diagram implies that for $\mathbf{P}$-almost every $\omega \in \Omega$, we have the following isomorphisms:

$$(\{\overline{\omega}\} \times X, \delta_{\overline{\omega}} \otimes \nu) \xrightarrow{\ \Psi\ } (\varphi^{-1}(\{\pi_C(\overline{\omega})\}), \eta_{\pi_C(\overline{\omega})}) \xrightarrow{\ \rho\ } (X, \nu)$$

Then for $\mathbf{P}$-almost every $\omega \in \Omega$ and ν-almost every $x \in X$, we have $\Psi(\overline{\omega}, x) = (\rho|_{\varphi^{-1}(\pi_C(\overline{\omega}))})^{-1}(x) \doteq \overline{\Psi}(\pi_C(\overline{\omega}), x)$. This implies that $\Psi : (B \times X, \nu_B \otimes \nu) \to (Y, \eta)$ factors through $\overline{\Psi} : (C \times X, \nu_C \otimes \nu) \to (Y, \eta)$ and that $(\varphi \otimes \rho) \circ \overline{\Psi} = \mathrm{id}_{C \times X}$ almost everywhere. $\square$

We now combine Theorem 5.19 and the induction procedure to prove Theorem 5.18.

Proof of Theorem 5.18. Set $X \doteq G/\Gamma$ and denote by $\nu \in \mathrm{Prob}(X)$ the unique G-invariant Borel probability measure on X. Choose a Borel section $\sigma : X \to G$ as in Corollary 1.12. Since $\Gamma < G$ is irreducible, the pmp action $G \curvearrowright (X, \nu)$ is irreducible. For every $g \in G$ and every $x \in X$, denote by $\tau(g, x) \in \Gamma$ the unique element in Γ such that $g\sigma(x) = \sigma(gx)\,\tau(g, x)$. Then $\tau : G \times X \to \Gamma$ is a Borel 1-cocycle.

Let $\Gamma \curvearrowright (Z, \zeta)$ be any nonsingular action and $\pi : (B, \nu_B) \to (Z, \zeta)$ any Γ-equivariant measurable factor map. Up to discarding a ν_B-null invariant measurable subset, we may assume that π is strictly Γ-equivariant (see [**Zi84**, Proposition B.5]). Define the induced space $\mathrm{Ind}_\Gamma^G(Z) = Z \times X$ and the induced action $G \curvearrowright \mathrm{Ind}_\Gamma^G(Z)$ by the formula

$$\forall g \in G, \forall x \in X, \forall z \in Z, \quad g \cdot (z, x) = (\tau(g, x)z, gx).$$

Define the G-equivariant measurable map

$$\Psi : B \times X \to \mathrm{Ind}_\Gamma^G(Z) : (b, x) \mapsto (\pi(\sigma(x)^{-1}b), x).$$

and set $\eta = \Psi_*(\nu_B \otimes \nu) \in \mathrm{Prob}(\mathrm{Ind}_\Gamma^G(Z))$. Observe that $\eta \sim \zeta \otimes \nu$. Define the G-equivariant measurable map

$$\rho : \mathrm{Ind}_\Gamma^G(Z) \to X : (z, x) \mapsto x.$$

We obtain the following (strictly) G-equivariant measurable factor maps

$$(B \times X, \nu_B \otimes \nu) \xrightarrow{\Psi} (\mathrm{Ind}_\Gamma^G(Z), \eta) \xrightarrow{\rho} (X, \nu)$$

such that $\rho \circ \Psi = p_X : B \times X \to X$.

Using Theorem 5.19, there exist a (G, μ)-boundary (C, ν_C) with its essentially unique G-equivariant measurable factor map $\pi_C : (B, \nu_B) \to (C, \nu_C)$ and a G-equivariant measurable isomorphism $\Phi : (\mathrm{Ind}_\Gamma^G(Z), \eta) \to (C \times X, \nu_C \otimes \nu)$ such that $\Phi \circ \Psi = \pi_C \otimes \mathrm{id}_X$ and $p_X \circ \Phi = \rho$ almost everywhere. We may choose conull G-invariant measurable subsets $Y_0 \subset \mathrm{Ind}_\Gamma^G(Z)$ and $Y_1 \subset C \times X$ so that $\Phi : Y_0 \to Y_1$ is measurable bijective and strictly G-equivariant, $p_X(\Phi(z, x)) = x$ for every $(z, x) \in Y_0$ and $\Phi(\Psi(b, x)) = (\pi_C(b), x)$ for every $(b, x) \in \Psi^{-1}(Y_0)$.

Define the measurable map $\varphi : Y_0 \to C$ such that for every $(z, x) \in Y_0$, we have $\Phi(z, x) = (\varphi(z, x), x)$. Then by G-equivariance, for every $g \in G$ and every $(z, x) \in Y_0$, we have

$$\varphi(\tau(g, x)z, gx) = g\varphi(z, x).$$

Then for every $(z, x) \in Y_0$, we have

$$(z, \Gamma) = (\tau(\sigma(x)^{-1}, x)z, \sigma(x)^{-1}x) = \sigma(x)^{-1} \cdot (z, x) \in Y_0$$

and $\varphi(z, \Gamma) = \sigma(x)^{-1}\varphi(z, x)$. Define the measurable subset $Z_0 = \{z \in Z \mid (z, \Gamma) \in Y_0\}$ and note that $Y_0 \subset Z_0 \times X$. Conversely, for every $z \in Z_0$ and every $x \in X$, we have $(z, x) = \sigma(x) \cdot (z, \Gamma) \in Y_0$. This shows that $Z_0 \times X \subset Y_0$. Thus, we have $Y_0 = Z_0 \times X$. Then $Z_0 \subset Z$ is ζ-conull. Moreover, for every $\gamma \in \Gamma$ and every $z \in Z_0$, we have $(\gamma z, \Gamma) = \gamma \cdot (z, \Gamma) \in Y_0$ and $\varphi(\gamma z, \Gamma) = \gamma\varphi(z, \Gamma)$. This implies that $Z_0 \subset Z$ is Γ-invariant and the measurable map $\psi : Z_0 \to C : z \mapsto \varphi(z, \Gamma)$ is Γ-equivariant. For every $(z, x) \in Z_0 \times X$, we have $\Phi(z, x) = (\sigma(x)\psi(z), x)$. Since Φ is injective, it follows that $\psi : Z_0 \to C$ is injective and so $\psi : Z_0 \to \psi(Z_0)$ is a measurable isomorphism.

For every $(b, x) \in \Psi^{-1}(Z_0 \times X)$, we have

$$(\sigma(x)\psi(\pi(\sigma(x)^{-1}b)), x) = \Phi(\pi(\sigma(x)^{-1}b), x) = \Phi(\Psi(b, x)) = (\pi_C(b), x).$$

Define the conull Γ-invariant measurable subset $B_0 = \pi^{-1}(Z_0) \subset B$. In particular, for every $b \in B_0$, we have $\Psi(b, \Gamma) = (\pi(b), \Gamma) \in Z_0 \times X$ and $\psi(\pi(b)) = \pi_C(b)$. This further implies that $\psi_*\zeta = \psi_*\pi_*\nu_B = \pi_{C*}\nu_B = \nu_C$. This finishes the proof of Theorem 5.18. $\square$

We now have all the tools available to prove Theorem 5.15.

Proof of Theorem 5.15. Firstly, assume that Γ/N is amenable. Let $i \in \{1, 2\}$. Since $N \lhd \Gamma$ is normal and since $p_i(\Gamma) < G_i$ is dense, $\overline{p_i(N)} \lhd G_i$ is a closed normal subgroup. Moreover, the well-defined group homomorphism $\Gamma/N \to G_i/\overline{p_i(N)}$ has dense range. Therefore, $G_i/\overline{p_i(N)}$ is amenable by Proposition 2.18.

Conversely, assume that for every $i \in \{1, 2\}$, $G_i/\overline{p_i(N)}$ is amenable. To prove that Γ/N is amenable, we use Theorem 2.20 and we show that $\ell^\infty(\Gamma/N)$ has a left invariant mean. Let $i \in \{1, 2\}$. Using Theorem 3.34, we may choose an admissible Borel probability measure $\overline{\mu}_i \in \mathrm{Prob}(G_i/\overline{p_i(N)})$ so that the $(G_i/\overline{p_i(N)}, \overline{\mu}_i)$-Poisson boundary is trivial. Choose an admissible Borel probability measure $\mu_i \in \mathrm{Prob}(G_i)$ so that $\overline{\mu}_i$ is the pushforward measure of μ_i under the quotient map $G_i \to G_i/\overline{p_i(N)}$. Denote by (B_i, ν_{B_i}) the (G_i, μ_i)-Poisson boundary. Set $\mu = \mu_1 \otimes \mu_2 \in \mathrm{Prob}(G)$ and $(B, \nu_B) = (B_1 \times B_2, \nu_{B_1} \otimes \nu_{B_2})$. Then Corollary 5.17 implies that (B, ν_B) is the (G, μ)-Poisson boundary.

Consider the nonsingular action $\Gamma \curvearrowright (B, \nu_B)$. Denote by $\mathrm{L}^\infty(B)^N \subset \mathrm{L}^\infty(B)$ the Γ-invariant weak*-closed unital $*$-subalgebra of all N-invariant essentially bounded measurable functions.

Claim 5.20. We have $\mathrm{L}^\infty(B)^N = \mathbb{C}\mathbf{1}_B$.

Indeed, by Corollary 5.17 and Theorem 5.18, for every $i \in \{1, 2\}$, there exists a (G_i, μ_i)-boundary (C_i, ν_{C_i}) and there exists a Γ-equivariant weak*-continuous unital $*$-isomorphism $\mathrm{L}^\infty(B)^N \cong \mathrm{L}^\infty(C)$ where $(C, \nu_C) = (C_1 \times C_2, \nu_{C_1} \otimes \nu_{C_2})$. Since N acts trivially on $\mathrm{L}^\infty(B)^N$, it follows that for every $i \in \{1, 2\}$, $\overline{p_i(N)}$ acts trivially on (C_i, ν_{C_i}) and so (C_i, ν_{C_i}) is a $(G_i/\overline{p_i(N)}, \overline{\mu}_i)$-space. Since by construction the $(G_i/\overline{p_i(N)}, \overline{\mu}_i)$-Poisson boundary is trivial, Corollary 3.28 implies that the probability measure $\nu_{C_i} \in \mathrm{Prob}(C_i)$ is G_i-invariant. This further implies that the probability measure $\nu_C \in \mathrm{Prob}(C)$ is G-invariant. Since (C, ν_C) is a (G, μ)-boundary, Corollary 3.27 implies that (C, ν_C) is trivial. This further implies that $\mathrm{L}^\infty(B)^N = \mathrm{L}^\infty(C) = \mathbb{C}\mathbf{1}_B$.

By Corollary 3.33, the nonsingular action $\Gamma \curvearrowright (B, \nu_B)$ is amenable. Then there exists a Γ-equivariant projection $\Phi : \mathrm{L}^\infty(\Gamma \times B) \to \mathrm{L}^\infty(B)$. Observe that $\ell^\infty(\Gamma/N) = \ell^\infty(\Gamma)^N \subset \mathrm{L}^\infty(\Gamma \times B)^N$ and Claim 5.20 shows that

$\Phi(\mathrm{L}^\infty(\Gamma \times B)^N) \subset \mathrm{L}^\infty(B)^N = \mathbb{C}\mathbf{1}_B$. The restriction of Φ to $\ell^\infty(\Gamma/N)$ yields a left Γ-invariant mean $\mathfrak{m} : \ell^\infty(\Gamma/N) \to \mathbb{C} : F \mapsto \Phi(F \otimes \mathbf{1}_B)$. Therefore, Γ/N is amenable by Theorem 2.20. $\square$

3. Proof of Bader–Shalom's normal subgroup theorem

We combine Theorems 5.8 and 5.15 to prove Theorem 5.4.

Proof of Theorem 5.4. Let $\{e\} \neq N \lhd \Gamma$ be a nontrivial normal subgroup. We show that Γ/N is finite by proving that Γ/N has property (T) and is amenable.

Claim 5.21. For every $i \in \{1, 2\}$, $p_i(N) \neq \{e\}$ and so $G_i/\overline{p_i(N)}$ is compact.

By contradiction, up to permuting the indices, we may assume that $p_1(N) = \{e\}$. Then $N = \{e\} \times p_2(N)$, which implies that $p_2(N) \lhd G_2$ is a nontrivial closed normal subgroup. Since G_2 is just noncompact, it follows that $K_2 = G_2/p_2(N)$ is compact. Regard $\overline{\Gamma} = \Gamma/N$ as an irreducible discrete subgroup in $G_1 \times K_2$. Still denote by $p_1 : G_1 \times K_2 \to G_1$ (resp. $p_2 : G_1 \times K_2 \to K_2$) the canonical factor map. We claim that $p_1(\overline{\Gamma}) < G_1$ is discrete. Indeed, otherwise, there exists a sequence $(\overline{\gamma}_n)_{n\in\mathbb{N}}$ in $\overline{\Gamma}$ such that $p_1(\overline{\gamma}_n) \to e$ in G_1 and $p_1(\overline{\gamma}_n) \neq e$ for every $n \in \mathbb{N}$. Since K_2 is compact, up to taking a subsequence, we may assume that there exists $k_2 \in K_2$ such that $p_2(\overline{\gamma}_n) \to k_2$ in K_2. Then $\overline{\gamma}_n = (p_1(\overline{\gamma}_n), p_2(\overline{\gamma}_n)) \to (e, k_2)$ in $G_1 \times K_2$. Since $\overline{\Gamma} < G_1 \times K_2$ is discrete, we have that $k_2 \in p_2(\overline{\Gamma})$ and $\overline{\gamma}_n = (e, k_2)$ eventually. This shows that $p_1(\overline{\gamma}_n) = e$ eventually, which is a contradiction. Therefore, $p_1(\Gamma) = p_1(\overline{\Gamma}) < G_1$ is discrete. Since $p_1(\Gamma) < G_1$ is also dense, it follows that $G_1 = p_1(\Gamma)$ is discrete. This is a contradiction and finishes the proof of Claim 5.21. For every $i \in \{1, 2\}$, since G_i is just noncompact, it follows that $G_i/\overline{p_i(N)}$ is compact.

Claim 5.22. Every continuous homomorphism $\varphi : G \to \mathbb{C}$ that vanishes on N is identically zero.

By contradiction, assume that there exists a nonzero continuous homomorphism $\varphi : G \to \mathbb{C}$ that vanishes on N. We claim that for every $i \in \{1, 2\}$, $\varphi|_{G_i}$ is nonzero. Indeed, otherwise, up to permuting the indices, we may assume that $\varphi|_{G_2} = 0$. Then $p_1(N) < \ker(\varphi|_{G_1})$. Claim 5.21 further implies that $\ker(\varphi|_{G_1}) \lhd G_1$ is a cocompact closed normal subgroup. Since $\{0\}$ is the only compact subgroup of $\mathbb{C}$, this implies that $\ker(\varphi|_{G_1}) = G_1$ and so $\varphi|_{G_1} = 0$. Thus, we have $\varphi = 0$, which is a contradiction.

Let now $i \in \{1, 2\}$. Since $\varphi|_{G_i} \neq 0$, $\ker(\varphi|_{G_i}) \lhd G_i$ is a proper closed normal subgroup, hence $\ker(\varphi|_{G_i}) = \{e\}$ or $\ker(\varphi|_{G_i}) \lhd G_i$ is cocompact. Since $\{0\}$ is the only compact subgroup of $\mathbb{C}$, we necessarily have $\ker(\varphi|_{G_i}) = \{e\}$. Then $\varphi|_{G_i} : G_i \to \mathbb{C}$ is an injective continuous homomorphism. In particular, G_i is abelian. Using the structure theorem of locally compact abelian groups (see [HR79, Theorem VI.24.30]), there exist $n \in \mathbb{N}$ and a locally compact abelian group H that contains an open compact subgroup K and such that $G_i = \mathbb{R}^n \times H$. Since $\varphi|_H : H \to \mathbb{C}$ is injective and since $K < \ker(\varphi|_H)$, it follows that $K = \{e\}$ and so H is discrete. Since G_i is nondiscrete and just noncompact, we necessarily have $H = \{e\}$ and $n = 1$. Thus, $G_i = \mathbb{R}$ for every $i \in \{1, 2\}$. This contradicts the assumption that $(G_1, G_2) \neq (\mathbb{R}, \mathbb{R})$ and finishes the proof of Claim 5.22.

Using Claims 5.21 and 5.22, Theorem 5.8 implies that Γ/N has property (T). Using Claim 5.21, Theorem 5.15 implies that Γ/N is amenable. Therefore, Γ/N is finite by Proposition 2.27. $\square$

Acknowledgments

I am grateful to Narutaka Ozawa for insightful discussions regarding reduced 1-cohomology and for suggesting the proof of Theorem 4.22.

Bibliography

[BS04] U. Bader, Y. Shalom, *Factor and normal subgroup theorems for lattices in products of groups.* Invent. Math. **163** (2006), 415–454.

[BBHP20] U. Bader, R. Boutonnet, C. Houdayer, J. Peterson, *Charmenability of arithmetic groups of product type.* `arXiv:2009.09952`

[Be96] B. Bekka, *On uniqueness of invariant means.* Proc. Amer. Math. Soc. **126** (1998), 507–514.

[BHV08] B. Bekka, P. de la Harpe, A. Valette, *Kazhdan's property (T).* New Mathematical Monographs, **11**. Cambridge University Press, Cambridge, 2008. xiv+472 pp.

[Be09] Y. Benoist, *Five lectures on lattices in semisimple Lie groups.* Géométries à courbure négative ou nulle, groupes discrets et rigidités, 117–176, Sémin. Congr., **18**, Soc. Math. France, Paris, 2009.

[BHC61] A. Borel, Harish-Chandra, *Arithmetic subgroups of algebraic groups.* Ann. of Math. **75** (1962), 485–535.

[BM00a] M. Burger, S. Mozes, *Groups acting on trees: from local to global structure.* Inst. Hautes Études Sci. Publ. Math. **92** (2000), 113–150.

[BM00b] M. Burger, S. Mozes, *Lattices in product of trees.* Inst. Hautes Études Sci. Publ. Math. **92** (2001), 151–194.

[Ca71] J.W.S. Cassels, *An introduction to the geometry of numbers.* Corrected reprint of the 1971 edition. Classics in Mathematics. Springer-Verlag, Berlin, 1997. viii+344 pp.

[CD60] G. Choquet, J. Deny, *Sur l'équation de convolution* $\mu = \mu * \sigma$. C. R. Acad. Sci. Paris **250** (1960), 799–801.

[DE14] A. Deitmar, S. Echterhoff, *Principles of harmonic analysis.* Second edition. Universitext. Springer, Cham, 2014. xiv+332 pp.

[De76] P. Delorme, 1-*cohomologie des représentations unitaires des groupes de Lie semi-simples et résolubles. Produits tensoriels continus de représentations.* Bull. Soc. Math. France **105** (1977), 281–336.

[EO16] A. Erschler, N. Ozawa, *Finite-dimensional representations constructed from random walks.* Comment. Math. Helv. **93** (2018), 555–586.

[Fu62a] H. Furstenberg, *A Poisson formula for semi-simple Lie groups.* Ann. of Math. **77** (1963), 335–386.

[Fu62b] H. Furstenberg, *Non commuting random products.* Trans. Amer. Math. Soc. **108** (1963), 377–428.

[Gu72] A. Guichardet, *Sur la cohomologie des groupes topologiques.* II. Bull. Sci. Math. **96** (1972), 305–332.

[HP06] U. Haagerup, A. Przybyszewska, *Proper metrics on locally compact groups, and proper affine isometric actions on Banach spaces.* `arXiv:math/0606794`

[HR79] E. Hewitt, K.A. Ross, *Abstract harmonic analysis. Vol.* I. Structure of topological groups, integration theory, group representations. Second edition. Grundlehren der Mathematischen Wissenschaften, **115**. Springer-Verlag, Berlin-New York, 1979. ix+519 pp.

[HS53] E. Hewitt, L.J. Savage, *Symmetric measures on Cartesian products.* Trans. Amer. Math. Soc. **80** (1955), 470–501.

[HM77] R.E. Howe, C.C. Moore, *Asymptotic properties of unitary representations.* J. Functional Analysis **32** (1979), 72–96.

[KV82] V.A. Kaimanovich, A.M. Vershik, *Random walks on discrete groups: boundary and entropy.* Ann. Probab. **11** (1983), 457–490.

[Ka67] D.A. Kazhdan, *On the connection of the dual space of a group with the structure of its closed subgroups.* Funkcional. Anal. i Priložen. **1** (1967) 71–74.

[KS96] N.J. Korevaar, R.M. Schoen, *Global existence theorems for harmonic maps to non-locally compact spaces.* Comm. Anal. Geom. **5** (1997), 333–387.

[LMR96] A. Lubotzky, S. Mozes, M.S. Raghunathan, *The word and Riemannian metrics on lattices of semisimple groups.* Inst. Hautes Études Sci. Publ. Math. **91** (2000), 5–53.

[Ma91] G.A. Margulis, *Discrete subgroups of semisimple Lie groups.* Ergebnisse der Mathematik und ihrer Grenzgebiete (3), **17**. Springer-Verlag, Berlin, 1991. x+388 pp.

[Mo65] C.C. Moore, *Ergodicity of flows on homogeneous spaces.* Amer. J. Math. **88** (1966), 154–178.

[Mo95] N. Mok, *Harmonic forms with values in locally constant Hilbert bundles.* Proceedings of the Conference in Honor of Jean-Pierre Kahane (Orsay, 1993). J. Fourier Anal. Appl. 1995, Special Issue, 433–453.

[NZ00] A. Nevo, R.J. Zimmer, *A generalization of the intermediate factors theorem.* J. Anal. Math. **86** (2002), 93–104.

[Ro81] J. Rosenblatt, *Ergodic and mixing random walks on locally compact groups.* Math. Ann. **257** (1981), 31–42.

[Ru87] W. Rudin, *Real and complex analysis.* Third edition. McGraw-Hill Book Co., New York, 1987. xiv+416 pp.

[Ru91] W. Rudin, *Functional analysis.* Second edition. International Series in Pure and Applied Mathematics. McGraw-Hill, Inc., New York, 1991. xviii+424 pp.

[Sh99] Y. Shalom, *Rigidity of commensurators and irreducible lattices.* Invent. Math. **141** (2000), 1–54.

[Sh03] Y. Shalom, *Harmonic analysis, cohomology, and the large-scale geometry of amenable groups.* Acta Math. **192** (2004), 119–185.

[St73] R.A. Struble, *Metrics in locally compact groups.* Compositio Math. **28** (1974), 217–222.

[Ta03] M. Takesaki, *Theory of operator algebras.* III. Encyclopaedia of Mathematical Sciences, **127**. Operator Algebras and Non-commutative Geometry, 8. Springer-Verlag, Berlin, 2003. xxii+548 pp.

[Zi76] R.J. Zimmer, *Amenable ergodic group actions and an application to Poisson boundaries of random walks.* J. Functional Analysis **27** (1978), 350–372.

[Zi84] R.J. Zimmer, *Ergodic theory and semisimple groups.* Monographs in Mathematics, **81**. Birkhäuser Verlag, Basel, 1984. x+209 pp.